MATHEMATICAL
PROBLEM PAPERS

CAMBRIDGE UNIVERSITY PRESS
London: FETTER LANE, E.C.
C. F. CLAY, Manager

Edinburgh: 100, PRINCES STREET
Berlin: A. ASHER AND CO.
Leipzig: F. A. BROCKHAUS
New York: G. P. PUTNAM'S SONS
Bombay and Calcutta: MACMILLAN AND CO., Ltd.
Toronto: J. M. DENT AND SONS, Ltd.
Tokyo: THE MARUZEN-KABUSHIKI-KAISHA

MATHEMATICAL
PROBLEM PAPERS

COMPILED AND ARRANGED

BY THE

REV. E. M. RADFORD, M.A.

MATHEMATICAL LECTURER AT ST JOHN'S COLLEGE, BATTERSEA,

FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE

Cambridge:

at the University Press

1914

First Edition 1904
Second Edition 1914

PREFACE TO FIRST EDITION

THIS collection of Problem Papers is intended primarily for the use of Candidates for Mathematical Entrance Scholarships at Oxford and Cambridge, but I hope that they will also be of service to students preparing for other Mathematical examinations. The book is divided into two parts, each containing fifty Papers, graduated in order of difficulty. Each paper contains questions on the usual Scholarship subjects, viz. Pure Geometry, Algebra, Trigonometry, Analytical Conics and Elementary Mechanics, while the second set of Papers also includes elementary questions on the Theory of Equations and the Differential Calculus.

A large number of the questions are original, a few have been communicated by friends, and the remainder are taken almost entirely from the Papers set at the various University and College Examinations, other than Entrance Scholarship Papers.

My thanks are due to Mr F. E. Jelly, M.A., of St John's School, Leatherhead, and to my colleague, Mr T. Ayres, M.Sc., for their kindness in reading the proof-sheets, and verifying the results of a large number of the original questions.

I hope to publish a volume of Solutions as soon as possible, and shall be very grateful for any solutions which may be sent me by masters and others using the book.

E. M. RADFORD.

St John's College,
Battersea,
July, 1904.

PREFACE TO SECOND EDITION

IN this Edition, the Papers have been thoroughly revised, and a number of errors in the results corrected. A large number of new questions have been inserted, replacing others which have been proved to be either too difficult or of no great interest.

Part II now contains questions on the Integral, as well as the Differential, Calculus. Students who include the Calculus in their course will find it advisable to attempt some of the Papers in this Part before those in Part I.

I am glad to be able to announce that a complete volume of Solutions to the new Edition is now in the press, and will shortly be published.

E. M. RADFORD.

St John's College,
Battersea,
March, 1914.

MATHEMATICAL PROBLEMS

PART I.

I.

1. Through a given point draw a straight line which shall form with two given straight lines a triangle of given perimeter.

2. Any radius of a circle is drawn and a circle is described upon it as diameter. Prove geometrically that the locus of the centre of a circle described so as to touch the large circle internally and the small circle externally is an ellipse and find the position of its foci and centre. Find also the magnitude of its eccentricity and the lengths of its axes and latus-rectum.

3. Resolve into five factors the expression

$$a^5(b-c)+b^5(c-a)+c^5(a-b)+abc(b-c)(c-a)(a-b).$$

4. Apply the Binomial Theorem to shew that

$$\left(\frac{3}{4}\right)^{\frac{4}{5}} = \cdot 7944 \text{ approximately.}$$

5. Shew that

$$\sin^2 12^\circ+\sin^2 21^\circ+\sin^2 39^\circ+\sin^2 48^\circ = 1+\sin^2 9^\circ+\sin^2 18^\circ.$$

6. $A_1A_2 \ldots A_n$ is a regular polygon of n sides and P any point between A_1 and A_n on its circumcircle which is of radius R. If PA_1 subtend an angle 2α at the centre, shew that the sum of the chords PA_1, PA_2, ... PA_n is

$$2R\left(\cos\alpha\cot\frac{\pi}{2n}+\sin\alpha\right).$$

7. Shew that the equation to the circumcircle of the triangle formed by the lines $y=\pm kx$ and $x\cos\alpha+y\sin\alpha-p=0$ is

$$(\cos^2\alpha-k^2\sin^2\alpha)(x^2+y^2)-p(1+k^2)(x\cos\alpha-y\sin\alpha)=0.$$

8. PQ is a double ordinate of a parabola, and the line joining P to the foot of the directrix cuts the curve in P'. Shew that $P'Q$ passes through the focus.

9. A balance has its arms unequal in length and weight. A certain article appears to weigh Q_1 or Q_2 according as it is put into one scale-pan or the other. Similarly another article appears to weigh R_1 or R_2. Shew that the true weight of an article which appears to weigh the same in whichever scale-pan it is put is

$$\frac{Q_1R_2 - Q_2R_1}{Q_1 - Q_2 - R_1 + R_2}.$$

10. A square lamina rests in a vertical plane perpendicular to a smooth vertical wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square. Shew that in the position of equilibrium the inclination of the string to the wall is $\cot^{-1} 3$.

11. A smooth wedge is placed on a smooth table, the principal vertical section of the wedge being a right-angled triangle, whose hypotenuse is inclined to the horizontal at an angle a. A string passing over a pulley at the top of the wedge connects a mass m hanging freely with a mass m' on the plane. If the mass m descend, prove that in order to keep the wedge from sliding a horizontal force will be required equal to

$$\frac{m'(m - m' \sin a) \cos a}{m + m'} g.$$

12. Prove that the greatest range of a particle, projected with a given velocity, on a given inclined plane, is four times the greatest vertical altitude above the inclined plane.

II.

1. A given point D lies between two given lines AB and AC. Find a construction for a line through D terminated by AB and AC, such that D is one of its points of trisection. Prove also that there are two such lines.

2. If a conic circumscribe a parallelogram, its centre must be at the intersection of the diagonals.

3. Prove the identity

$$\frac{a^2(b-c)}{b+c-a}+\frac{b^2(c-a)}{c+a-b}+\frac{c^2(a-b)}{a+b-c}+\frac{(a+b+c)^2(b-c)(c-a)(a-b)}{(b+c-a)(c+a-b)(a+b-c)}=0.$$

4. If there be any number of quantities $a, b, c, \ldots$, shew that

$$a^3+b^3+c^3+\ldots-3(abc+abd+\ldots)$$

is divisible by $a+b+c+\ldots$ and find the quotient.

5. If in a triangle a, c and C are given, and b_1, b_2 are the two values of the third side, and r_1, r_2 the radii of the two inscribed circles, prove that

$$\text{(i)}\quad \left(\frac{b_1}{r_1}-\cot\tfrac{1}{2}C\right)\left(\frac{b_2}{r_2}-\cot\tfrac{1}{2}C\right)=1.$$

$$\text{(ii)}\quad r_1r_2=a(a-c)\sin^2\tfrac{1}{2}C.$$

6. Prove that if $\cos A=\cos\theta\sin\phi$, $\cos B=\cos\phi\sin\psi$, $\cos C=\cos\psi\sin\theta$, and $A+B+C=\pi$, then

$$\tan\theta\tan\phi\tan\psi=1.$$

7. Prove that the locus of the poles of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the fixed point (h, k) is the circle

$$(h^2+k^2-a^2)(x^2+y^2)-2a^2hx-2a^2ky+2a^4=0.$$

8. Chords of the parabola $y^2 = 4ax$ are drawn through the fixed point (h, k). Shew that the locus of their middle points is the parabola

$$y(y-k) = 2a(x-h).$$

9. P and Q are extremities of two conjugate diameters of an ellipse of minor axis $2b$, and S is a focus. Prove that

$$PQ^2 - (SP - SQ)^2 = 2b^2.$$

10. A rod of length $2a$ rests on a smooth vertical circle of radius b, one end being attached to a string, to the other end of which is tied a weight which hangs down over the circle. The distance of the point of contact from this end of the rod is na. Prove that in the position of equilibrium the rod makes with the vertical an angle

$$\tan^{-1}\left\{\frac{2abn^2}{(1-n)(b^2-a^2n^2)}\right\}.$$

11. The base angles of a wedge are α and β and its mass is M. Two particles of masses m and m' are let fall simultaneously from the vertex down the two faces. Prove that the wedge will move on the smooth horizontal plane with which it is in contact with acceleration

$$\tfrac{1}{2} g(m \sin 2\alpha - m' \sin 2\beta)/(M + m \sin^2 \alpha + m' \sin^2 \beta).$$

12. If the unit of kinetic energy be that of a train of mass m tons moving with a velocity of v miles an hour, the unit of power that of an engine of horse-power h, and the unit of force the weight of n tons, prove that the unit of mass is

$$\frac{1}{2} m \left(\frac{448}{75} \frac{vn}{h}\right)^2 \text{ tons.}$$

III.

1. A is the centre of a circle, and B is any point outside it. BC is a straight line drawn from B to cut the circle at C, and BD, BE are the tangents from B. Through C a straight line is drawn perpendicular to BC to cut AD in T, and AE in U. Prove that AB is a mean proportional between AT and AU.

2. PGQ is a chord of a parabola meeting the axis in G. Prove that the distance of G from the vertex, the ordinates of P and Q and the latus-rectum are proportionals.

3. Shew that the coefficient of x^{n-2} in the expansion of $(1-\frac{3}{4}x)^{-\frac{11}{3}}$ is

$$\frac{2 \,.\, 5 \,.\, 8 \ldots (3n+2)}{5 \,.\, 4^n \,.\, (n-2)\,!}.$$

4. Shew that the 42nd power of any number which is not a multiple of 7 is of the form $49m+1$.

5. Prove that the ratio of the area of the triangle formed by the points of contact of the inscribed circle of a triangle ABC to the area of ABC is half the ratio of the radii of the inscribed and circumscribed circles of the triangle ABC.

6. Find approximately, by a graphical method, the circular measure of the acute angle which satisfies the equation

$$\cot x = 4x.$$

7. Shew that the locus of the poles of tangents to the circle $x^2+y^2=a^2$ with respect to the circle $x^2+y^2=2bx$ is the conic

$$(a^2-b^2)\,x^2+a^2y^2-2a^2bx+a^2b^2=0.$$

8. If tangents be drawn from points on the line $x=c$ to the parabola $y^2=4ax$, shew that the locus of the intersection of the corresponding normals is the parabola

$$ay^2=c^2\,(x+c-2a).$$

9. Tangents are drawn from any point on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=4$ to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$: prove that they form with their chord of contact a triangle whose centroid is on the latter conic.

10. The faces AB, AC of a fixed triangular wedge make angles β, γ with the horizon. Equal weights connected by a string which passes without friction over the vertex A rest on the faces, the one weight being just on the point of moving up, the other of moving down. Shew that $\mu=\tan\frac{1}{2}(\beta-\gamma)$, where μ is the coefficient of friction for either weight.

11. If the radii of a wheel and axle are 1 ft. and 3 in., and the suspended weights are 1 lb., 3 lb. respectively, prove that the heavier weight will ascend with acceleration $\dfrac{1}{19}g$, the inertia of the machine being neglected.

12. It is required to throw a projectile from a given point below a given plane whose inclination to the horizon is α, so as to strike the plane. Shew that the velocity of projection must exceed $(2gc\cos\alpha)^{\frac{1}{2}}$, where c is the distance of the point from the plane.

IV.

1. AB, AC are two radii of a circle inclined at an angle of 60°. Upon AC a point P is taken such that a circle can be described with centre P to touch the circle internally and also to touch a circle on AB as diameter externally. Prove that

$$AP = \tfrac{4}{5} AC.$$

2. Given a directrix and two points on a conic, find the locus of the corresponding focus.

3. Prove that if $u = a^2 + ab + b^2$ and $v = a^2 - ab + b^2$, then

$$4(a^4 + b^4) = 6uv - u^2 - v^2,$$
$$4(a^6 + b^6) = 3u^2v + 3uv^2 - u^3 - v^3.$$

4. If $x < 1$, shew that

$$\frac{x+2}{x^2+x+1} = 2 - x - x^2 + 2x^3 - x^4 - x^5 + \ldots.$$

5. Shew that

$$\cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15} + \cos\frac{14\pi}{15} = \frac{1}{2}.$$

6. The inscribed circle of a triangle ABC touches the sides at D, E, F, and I is its centre. Circles whose radii are ρ_1, ρ_2, ρ_3 are inscribed in the quadrilaterals $IEAF$, $IFBD$, $IDCE$: prove that

$$\frac{\rho_1}{r-\rho_1} + \frac{\rho_2}{r-\rho_2} + \frac{\rho_3}{r-\rho_3} = \frac{\rho_1\rho_2\rho_3}{(r-\rho_1)(r-\rho_2)(r-\rho_3)}.$$

7. A straight line parallel to $lx + my = 1$ meets the axes (supposed inclined at an angle ω) in P and Q. From P and Q perpendiculars are drawn to the lines $py + x = 0$, $qx + y = 0$ respectively. Shew that these perpendiculars meet on a fixed straight line, and find its equation.

8. Two tangents t_1 and t_2 are drawn to a parabola; h is the internal bisector of the angle between them and t is the tangent parallel to h. Shew that the product of the perpendiculars from the focus on t_1 and t_2 is the same as that on t and h.

9. If the points of intersection of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ are at the extremities of conjugate diameters of the former, prove that

$$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=2.$$

10. The centre of gravity of a quadrilateral area is the same as that of four equal particles placed at its angular points: shew that the quadrilateral must be a parallelogram.

11. The masses in an Atwood's machine are 5 lbs. and 3 lbs., and the machine is being used in a lift which is ascending uniformly with acceleration 2 ft. per sec. per sec. Find the accelerations of the two masses and the tension of the string.

12. A particle projected with a given velocity from a point on a horizontal plane strikes normally at P a vertical wall at distance a. Shew that if b is the vertical distance between the possible positions of P, and c is the height to which the particle would have ascended if projected vertically, then

$$a^2+b^2=c^2.$$

V.

1. If $AA'BB'$, $BB'CC'$, $CC'AA'$ are three circles, and the straight lines AA', BB', CC' cut the circle $A'B'C'$ in α, β, γ respectively, then the triangle $\alpha\beta\gamma$ will be similar to ABC.

2. If the diagonals of a quadrilateral circumscribing a conic intersect in a focus, they are at right angles, and the third diagonal is the corresponding directrix.

3. If n be any integer, and if the expansion of $(1+x)^{2n}$ be $1 + a_1x + a_2x^2 + a_3x^3 + \ldots$, prove that the sum of n terms of the series $a_2 - a_3 + a_4 - a_5 + \ldots$ is

$$2n - 1 + \frac{(-1)^{n+1}.(2n-1)!}{(n+1)!\,(n-2)!}.$$

4. Prove that the problem of dividing a solid hemisphere of unit radius into two equal parts by means of a plane parallel to the base involves the solution of the equation

$$x^3 - 3x + 1 = 0,$$

and find graphically the root of this equation which is applicable to the problem in question.

5. If $A + B + C = \pi$, $\gamma - \beta = \pi - A$, $\beta - \alpha = \pi - C$, prove that
$\sin A \cos 2\alpha + \sin B \cos 2\beta + \sin C \cos 2\gamma$

$$+ (\sin A + \sin B + \sin C)(\cos \overline{\beta + \gamma} + \cos \overline{\gamma + \alpha} + \cos \overline{\alpha + \beta}) = 0.$$

6. On the sides of a triangle as bases are described internally three isosceles triangles with base angles θ. If the triangle formed by their vertices is similar to the original triangle, then

$$\tan\theta = \frac{\sin A \sin B \sin C}{1 + \cos A \cos B \cos C}.$$

7. From a point $P(x', y')$ perpendiculars PM, PN are drawn on the straight lines $ax^2 + 2hxy + by^2 = 0$. Shew that if O is the origin, the area of the triangle OMN is

$$\frac{(ay'^2 - 2hx'y' + bx'^2)(h^2 - ab)^{\frac{1}{2}}}{(a-b)^2 + 4h^2}.$$

8. If the normal at P to a parabola meet the curve again in P' and the normals at P, P' make angles α, α' with the axis, prove that

$$3\cos\alpha' + \cos(2\alpha - \alpha') = 0.$$

9. Shew that the four common tangents of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the rectangular hyperbola $xy = c^2$ are the lines

$$\{4c^2x^2 + a^2(2xy - 4c^2)\}\{4c^2y^2 + b^2(2xy - 4c^2)\} + (b^2x^2 - a^2y^2)^2 = 0.$$

10. A cylindrical ruler of radius a and length $2l$ rests on a horizontal rail with one end against a smooth vertical wall to which the rail is parallel. Shew that the angle the axis of the ruler makes with the vertical is given by

$$(l\sin\theta + a\cos\theta)\sin^2\theta + 2a\cos\theta = b,$$

where b is the distance of the rail from the wall.

11. An elastic ball is dropped from the ceiling of a room of height h, and at the same instant a similar ball is dropped from a point midway between the ceiling and the floor. If the coefficient of restitution in each case is e, shew that the balls will first pass each other at a height

$$\frac{4e^2 + 4e - 1}{8(1+e)^2}h$$

from the floor.

12. A smooth wedge, whose section is an isosceles right-angled triangle, rests on a perfectly smooth table, the hypotenuse being in contact with the table. Two masses m_1, m_2 are attached to the ends of a string, which passes over a very small pulley at the summit of the wedge. Prove that the wedge moves on the table with acceleration

$$\frac{(m_1 - m_2)g}{2M + m_1 + m_2},$$

where M is the mass of the wedge.

VI.

1. Two circles are drawn touching the sides AB, AC of a triangle ABC at the ends of the base BC and also passing through its middle point D. If E be the other point of intersection, prove that $DA \cdot DE = DC^2$.

2. If a line be drawn through a focus of a central conic, making a constant angle with a tangent, prove that the locus of the point of intersection is a circle.

3. Shew that, if n be a positive integer,

$$(x+y+z)^{2n+1} - x^{2n+1} - y^{2n+1} - z^{2n+1}$$

is, save for a numerical factor, divisible by

$$(x+y+z)^3 - x^3 - y^3 - z^3.$$

4. Shew that the expression

$$\sum_{r=1}^{r=n} \frac{a_r^{m-1}}{(a_r - a_1)(a_r - a_2) \dots (a_r - a_{r-1})(a_r - a_{r+1}) \dots (a_r - a_n)}$$

is equal to zero if $m < n$, to unity if $m = n$ and to $a_1 + a_2 + \dots + a_n$ if $m = n + 1$.

5. By means of a circle and the cosine-graph, or otherwise, find approximately the positive root of the equation

$$x^2 + \cos^2 x = 2.$$

6. Prove that

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}.$$

7. Prove, without assuming anything but the definition, that the straight lines

$$3x - y = 0 \text{ and } x + 17y = 0$$

are conjugate diameters of the conic

$$2x^2 - xy + 3y^2 = 1.$$

8. The tangent at a point P of a parabola meets another parabola with the same vertex and axis in Q and R. Shew that, if θ_1, θ_2, θ_3 are the angles the tangents at P, Q, R make with the common axis, then $\cot\theta_1$ is the arithmetic mean between $\cot\theta_2$ and $\cot\theta_3$.

9. The tangent at P to the hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ meets the lines $y^2=c^2$ at the points Q and R. Shew that c can be chosen so that CP bisects the angle QCR for all positions of P on the curve.

10. A weight W is attached to an endless string of length l, which hangs over two smooth pegs distant c apart in a horizontal line. Prove that the pressure on each peg is of magnitude

$$W\{(l-c)/2(l-2c)\}^{\frac{1}{2}}.$$

11. A piece of wire of given length is bent into the form of a circular quadrant and its two bounding radii: find the position of its centre of gravity.

12. Two heavy particles are projected from a point with equal velocities, their directions of projection being in the same vertical plane: t, t' are the times taken by them to reach the point of intersection of their paths, and T, T' the times to reach their highest point. Shew that $Tt+T't'$ is independent of the directions of projection.

VII.

1. From a point M in the side BC of a triangle ABC lines MB', MC' are drawn parallel to AC, AB meeting AB, AC in B' and C'. The lines BC', CB' intersect in P and AP intersects $B'C'$ in M'. Prove that

$$M'B' : M'C' = MB : MC.$$

2. T is any point on the tangent to a parabola at Q. Prove that the tangent at T to the circle round TQS touches the parabola.

3. Find the relations connecting the roots of two quadratic equations

$$ax^2 + 2bx + c = 0, \quad a'x^2 + 2b'x + c' = 0$$

which correspond to the relations

(i) $ac' + a'c - 2bb' = 0$,

(ii) $(ac' + a'c - 2bb')^2 - 4(ac - b^2)(a'c' - b'^2) = 0$.

4. Sum the following series:

(i) $1 . 2 . 4 + 2 . 3 . 5 + 3 . 4 . 6 + \ldots$ to n terms,

(ii) $2 + 5 + 12 + 31 + 86 + 249 + \ldots$ to n terms,

(iii) $1 + \dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$ to infinity (where $b > a + 1$).

5. Shew that

$$8 \sin(A+B) \sin(B+C) \sin(C+A) \sin(A-B) \sin(B-C) \sin(C-A) + \Sigma \cos 4A \sin(B-C) \sin(B+C) = 0.$$

6. If A, B, C be the angles of a triangle, prove that

$$\begin{vmatrix} \cos(A+2B) & \cos B & \cos C \\ \cos A & \cos(B+2C) & \cos C \\ \cos A & \cos B & \cos(C+2A) \end{vmatrix} = 0.$$

7. Prove that the equation to the line joining the middle points of the diagonals of the quadrilateral formed by the lines

$$ax^2 + 2hxy + by^2 = 0, \quad a'(x-\alpha)^2 + 2h'y(x-\alpha) + b'y^2 = 0$$

is

$$(ah' - a'h)(2x - \alpha) + (ab' - a'b)y = 0.$$

8. Through each point of the straight line $y = mx + c$ is drawn a chord of the parabola $y^2 = 4ax$ which is bisected at the point. Shew that these chords touch the parabola

$$\left(y + \frac{2a}{m}\right)^2 = 8a\left(x + \frac{c}{m}\right).$$

9. If a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, shew that the locus of its middle point is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

10. A uniform beam, of weight W, can move freely in a vertical plane about a hinge at one end A. To the other end B a string is fastened which passes over a small fixed smooth pulley vertically above A, and supports a weight w at the other end: prove that the beam can rest inclined to the horizontal at an angle whose sine is

$$\frac{l}{2a}\left(1 - \frac{4w^2}{W^2}\right) + \frac{a}{2l},$$

where a is the length of the beam, and l the height of the pulley above A.

11. A particle of mass m slides down the rough upper face of a wedge of mass M and angle α. The lower face of the wedge is smooth, and can move on a smooth horizontal plane. Shew that its acceleration is

$$\frac{m \cos\alpha \sin(\alpha - \epsilon)}{M\cos\epsilon + m\sin\alpha\sin(\alpha - \epsilon)} g,$$

where ϵ is the angle of friction.

12. If t and t' be the times of flight of a particle corresponding to a given range on an inclined plane of angle α, shew that

$$t^2 + t'^2 + 2tt' \sin\alpha = \frac{4u^2}{g^2},$$

where u is the velocity of projection in each case.

VIII.

1. Three equal circles are drawn each touching the other two externally, and a fourth circle is drawn touching each of the former externally. If tangents be drawn from any point on the latter circle to the three former, shew that one of the three is equal to the sum of the other two.

2. The tangents to a parabola at P and P' meet in Q, and the circumcircles of the triangles SPQ, $SP'Q$ meet the axis again in R and R'. Prove that PR and QR' are parallel.

3. If $a+b+c+d=0$, $a^2+b^2+c^2+d^2=0$, shew that

$$a^8+b^8+c^8+d^8=\tfrac{1}{4}(a^4+b^4+c^4+d^4)^2.$$

4. Simplify the expression

$$\frac{(a-x)(a-y)(a-z)}{a(a-b)(a-c)(a-d)}+\frac{(b-x)(b-y)(b-z)}{b(b-a)(b-c)(b-d)}$$
$$+\frac{(c-x)(c-y)(c-z)}{c(c-a)(c-b)(c-d)}+\frac{(d-x)(d-y)(d-z)}{d(d-a)(d-b)(d-c)}.$$

5. If $lm+mn+nl=1$, deduce from trigonometrical considerations that

$$\frac{1}{mn(1+l^2)}+\frac{1}{nl(1+m^2)}+\frac{1}{lm(1+n^2)}$$
$$=\frac{2}{lmn(1+l^2)^{\frac{1}{2}}(1+m^2)^{\frac{1}{2}}(1+n^2)^{\frac{1}{2}}}.$$

6. If the equations

$$x\cos\beta+y\cos\alpha=c \text{ and } x^2-2xy\cos\gamma+y^2=c^2$$

have a unique solution, shew that one of the angles $\pm\alpha\pm\beta\pm\gamma$ must be an odd multiple of π.

7. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent two straight lines and parallels be drawn to them through the origin, shew that the area of the parallelogram so formed is

$$\frac{c}{2\sqrt{h^2 - ab}}.$$

8. Find the equation to the axis and the length of the latus-rectum of the parabola

$$x^2 - 2xy + y^2 - 6x - 10y + 9 = 0$$

and draw the curve.

9. The rectangular hyperbola $xy = c^2$ is cut by a circle passing through its centre C in four points P_1, P_2, Q_1, Q_2. Prove that if p and q be the perpendiculars from C on the chords P_1P_2 and Q_1Q_2, then

$$pq = c^2.$$

10. AB, BC, CD are three equal rods of no appreciable weight, smoothly hinged together at B and C, and to fixed points at A and D, the figure forming one half of a regular hexagon with BC horizontal and below AD. The framework is stiffened by another light rod AC, and loads of 10 and 30 lbs. hang from B and C respectively. Find graphically the stresses in the various rods.

11. An engine of mass M tons working at horse-power H draws n carriages each of mass M' tons at the uniform rate of v miles per hour. Supposing the resistance on the engine and on each carriage to be proportional to the weight, prove that the tension of the coupling between the engine and the nearest carriage is equal to the weight of

$$\frac{75}{448} \frac{HnM'}{(M + nM')v} \text{ tons.}$$

12. Shew that if in a conical pendulum which is making n revolutions per second, the length l of the string be diminished by a small length x, the increase in the number of revolutions per second is approximately $\frac{1}{2}\frac{nx}{l}$, the inclination of the string to the vertical remaining the same.

IX.

1. O is the orthocentre of a triangle ABC and K, L, M its images in the sides. Shew that the triangle KLM has the same circumcentre as ABC.

2. If the normal at a point P of a conic cut the axis in G and N be taken on the axis so that $NG = PG$, and NP' be the ordinate at N, then $P'N$ is the difference of the focal distances of P and P'.

3. Solve the equations:

$$\text{(i)}\quad \frac{x^2}{a+x} + \frac{y^2}{a+y} = a, \qquad x^2 + y^2 = 2a^2.$$

$$\text{(ii)}\quad x^2 + 2xy - 4x + y = 0, \quad y^2 - 2xy + 6x - 5y = 0.$$

4. Shew that

$$\begin{vmatrix} 2, & c+\frac{1}{c}, & b+\frac{1}{b} \\ c+\frac{1}{c}, & 2, & a+\frac{1}{a} \\ b+\frac{1}{b}, & a+\frac{1}{a}, & 2 \end{vmatrix} = \frac{2(bc-a)(ca-b)(ab-c)(abc-1)}{a^2b^2c^2}.$$

5. The angles of a triangle are determined from measurements of the sides which may err in excess or defect one per cent. of their value. Shew that if A is an obtuse angle, the greatest possible errors in A, B, C are

$$\frac{1}{50}\sin A \operatorname{cosec} B \operatorname{cosec} C, \quad \frac{1}{50}\cot C, \quad \frac{1}{50}\cot B,$$

in circular measure.

6. If α, β, γ be three angles, such that

$$\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0,$$

shew that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \tfrac{3}{2}.$$

7. If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of straight lines, shew that the area of the triangle formed by the bisectors of the angles between them and the axis of x is

$$\frac{\sqrt{(a-b)^2 + 4h^2}}{2h} \cdot \frac{ca - g^2}{ab - h^2}.$$

8. Shew that the envelope of chords of the parabola $y^2 = 4ax$, which subtend an angle of 45° at the vertex, is the ellipse

$$x^2 + 8y^2 - 24ax + 16a^2 = 0.$$

9. Shew that the equation

$$17x^2 - 12xy + 8y^2 + 26x - 8y + 2 = 0$$

represents an ellipse whose axes are the lines

$$4x^2 - 4y^2 + 6xy + 7x + 4y + 3 = 0.$$

10. $ABCD$ is a cyclic quadrilateral: shew that if forces proportional to CD and CB act along AB and AD respectively, their resultant acts along AC and is proportional to BD.

11. An oscillating pendulum-bob is such that, in its lowest position, the tension of the string is n times the weight of the bob. Prove that the angle of swing on each side of the vertical is

$$\cos^{-1}\{\tfrac{1}{2}(3-n)\}.$$

12. Three particles are projected from the same point with velocities v_1, v_2, v_3, and the angles of projection are in Arithmetical Progression. The particles all strike the ground at the same point, and the first is at its highest position when the third strikes the ground. Prove that $\frac{\sqrt{5}}{2} v_2$ is a mean proportional between v_1 and v_3.

X.

1. Straight lines AD, BE, CF which meet in a point P are drawn from the angular points of a triangle to meet the opposite sides in D, E, F respectively. Prove that the straight lines joining A, B, C to the middle points of EF, FD, DE respectively meet in a point.

2. Given one tangent of a hyperbola, one focus and the eccentricity, find the locus of the other focus.

3. Shew that if $x+y+z$ is a factor of

$$ax^3+by^3+cz^3+dyz^2+d'y^2z+ezx^2+e'z^2x+fxy^2+f'x^2y+gxyz,$$

then $$a-b+f-f'=b-c+d-d'=c-a+e-e'=0,$$

and $$a+b+c=d'+e'+f'-g.$$

4. Find the turning values of the function

$$(x^2-x+1)/(x+2),$$

and draw the graph of the function.

5. Prove that

$$\sin 2x \sin 2y \sin 2z \sin (y-z) \sin (z-x) \sin (x-y)$$
$$+ \Sigma \sin 2x \sin (y-z) \cos^2 (y+z) \cos (x-y) \cos (x-z) \equiv 0.$$

6. If T_A, T_B, T_C be the tangents from the angular points A, B, C of a triangle to the inscribed circle of the pedal triangle, shew that

$$T_A^2 \sin A \sin (B-C) + T_B^2 \sin B \sin (C-A)$$
$$+ T_C^2 \sin C \sin (A-B) = 0.$$

7. Shew that if the sum of the squares of the perpendiculars from a point P on the lines $ax^2+2hxy+by^2=0$ is constant and equal to c^2, the locus of P is the conic

$$2(a^2-ab+2h^2)x^2+4(a+b)hxy$$
$$+2(b^2-ab+2h^2)y^2=c^2\{(a-b)^2+4h^2\}.$$

8. If tangents be drawn to the circle $x^2+y^2=a^2$ from points on the circle $x^2+y^2=2bx$, their chords of contact envelop the parabola

$$b^2y^2+a^2(2bx-a^2)=0.$$

9. A circle is drawn circumscribing the triangle formed by the chord $lx+my=1$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the tangents at its extremities. Shew that the equation to the other chord of intersection of the circle and the ellipse is

$$(lx-my)(a^2-b^2)=a^4l^2+b^4m^2.$$

10. A uniform straight rod rests with its lower end against a rough vertical wall, and is supported by a rough horizontal wire parallel to the wall. If α be the greatest inclination the rod can have to the wall when in equilibrium, prove that, when the rod rests in that position, the portion of it between the wall and the wire bears to the whole rod a ratio

$$\sin\alpha\sin(\alpha-2\lambda) : 1+\cos 2\lambda,$$

where λ is the angle of friction for the rod and both the wall and the wire.

11. A mass M rests on a smooth horizontal table: to it is attached a fine string which passes over a fixed smooth pulley in the same horizontal plane, and has a smooth pulley of mass μ hanging at the other end; over this second pulley a second fine string passes and hangs vertically, having masses m and m' attached to its ends. The system being in motion under the action of gravity, prove that the tension of the first string is

$$Mg\left\{1-\frac{M(m+m')}{(M+\mu)(m+m')+4mm'}\right\}.$$

12. A particle is projected from the foot of a double inclined plane, so as just to graze the top and then slide down the other face. Prove that the greatest height is $\frac{9}{8}$ that of the plane, and that if V is the velocity with which it strikes the plane, a the length and α the angle of the plane, then

$$V^2=\tfrac{1}{4}ag\operatorname{cosec}\alpha.$$

XI.

1. Let CD be a diameter of a circle, centre O; AB a chord perpendicular to CD, the point of intersection being M. On OM as diameter draw another circle. At any point T on this circle draw a tangent meeting the outer circle in E. Prove that $EA^2 + EB^2 = 4ET^2$.

2. Y and Z are the feet of the perpendiculars from a focus of an ellipse on a fixed and variable tangent respectively, and T is the point where the tangents meet. Prove that the locus of the point of intersection of YZ and the line joining T to the other focus is a circle through Y.

3. If the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ have one root common, and if the remaining root of the second is the square of the remaining root of the first, then

$$aa'bcc' + a^3c'^2 + c^3a'^2 = 0.$$

4. If a is a positive integer, sum the series

(i) $$1 - 2a + \frac{3a(a-1)}{2!} - \frac{4a(a-1)(a-2)}{3!} + \ldots,$$

(ii) $$1 - a^2 + \frac{a^2(a-1)^2}{1^2 . 2^2} - \frac{a^2(a-1)^2(a-2)^2}{1^2 . 2^2 . 3^2} + \ldots.$$

5. A tower of slant height a leans due N. and subtends angles ϕ_1, ϕ_2 at two points on a road running N.W. from its base. The distance between the points is b. Prove that the sine of the inclination of the tower to the vertical is

$$\frac{\sqrt{2}\,\{a^2 \sin^2(\phi_1 - \phi_2) - b^2 \sin^2\phi_1 \sin^2\phi_2\}^{\frac{1}{2}}}{a \sin(\phi_1 - \phi_2)}.$$

6. In any triangle, with the usual notation, prove that

$$2R^2(1 + \cos A \cos B \cos C) = \tfrac{1}{4}(a^2 + b^2 + c^2).$$

Prove also that each of these expressions is equal to the sum of the squares of the tangents drawn from the angular points of the triangle (one from each) to the nine-point circle.

7. If x_1, x_2, x_3 be three distinct solutions of the equation

$$\tan(\alpha+\beta-x)\tan(x+\beta-\alpha)\tan(x+\alpha-\beta)=1,$$

prove that $$x_1+x_2+x_3=n\pi+\left(\alpha+\beta+\frac{\pi}{4}\right).$$

8. Prove that the equation to the circle for which the triangle formed by the points $(a, 0)$, $(-b, 0)$, $(0, c)$ is self-conjugate is

$$c(x^2+y^2)-2aby+abc=0.$$

9. P and Q are points on the parabola $y^2=4a(x+a)$, such that the angle PSQ is of constant magnitude 2α. Shew that the point of intersection of the tangents at P and Q lies on the conic

$$y^2-\tan^2\alpha\,.\,x^2-4a\sec^2\alpha\,(x+a)=0.$$

Determine also the nature and eccentricity of this conic.

10. A uniform prism of square section $ABCD$ rests on a horizontal plane with AB lowest and a wedge-shaped piece is cut off by a plane parallel to the horizontal edges passing through the middle point of BC and cutting AB in F. Prove that when the wedge is removed the prism will fall over unless

$$AF>AB(\sqrt{6}-2).$$

11. In order to raise a weight which is half as much again as his own, a man fastens a rope to it and passes the rope over a smooth pulley: he then climbs up the rope with an acceleration $\frac{6}{7}g$, relative to the rope. Shew that the weight rises with acceleration $\frac{1}{7}g$, and find the tension of the rope.

12. A particle is projected from the lowest point of the roof of a house up the roof, and falls so as just not to strike the eaves on the other side. Shew that the velocity of projection is

$$\frac{1}{2}\left\{\frac{gl(1+8\sin^2\theta)}{\sin\theta}\right\}^{\frac{1}{2}},$$

where θ is the angle of elevation of the roof, and l the distance of its highest point from the eaves.

XII.

1. ABC is a triangle inscribed in a circle. The angles A, B, C are bisected by straight lines which meet the circle in A_1, B_1, C_1. If another triangle $A_2B_2C_2$ be similarly formed from $A_1B_1C_1$, and the process be continued indefinitely, prove that the triangles so formed tend to become equilateral.

2. A moving tangent to a parabola meets three fixed tangents in D, E, F. Prove that the ratios $EF : FD : DE$ are constant.

3. Shew that the result of eliminating x, y, z from the equations

$$x + y + z = a,$$
$$(y-z)^2 + (z-x)^2 + (x-y)^2 = b,$$
$$x(y-z)^2 + y(z-x)^2 + z(x-y)^2 = c,$$
$$x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2 = d,$$

is
$$b^2 - 2a^2b + 12ac - 18d = 0.$$

4. If 16 be added to the product of four consecutive odd or even numbers, the result is always a square number. For odd numbers its last digit in four cases out of five is 1, in the remaining case 5. For even numbers the last digit in four cases out of five is 6, in the remaining case 0.

5. If the bisectors of the angles of a triangle meet the opposite sides in D, E, F, shew that the area of the triangle DEF bears to that of the triangle ABC the ratio

$$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2}.$$

6. Prove that the roots of the equation

$$x^3 - 21x^2 + 35x - 7 = 0$$

are
$$\tan^2\frac{\pi}{7},\ \tan^2\frac{2\pi}{7},\ \tan^2\frac{3\pi}{7},$$

and hence shew that

$$\sec^4\frac{\pi}{7} + \sec^4\frac{2\pi}{7} + \sec^4\frac{3\pi}{7} = 416.$$

7. Prove that the distance of the origin from the orthocentre of the triangle formed by the lines

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1$$

and

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{(a+b)\,\alpha\beta\,(\alpha^2+\beta^2)^{\frac{1}{2}}}{a\alpha^2 - 2h\alpha\beta + b\beta^2}.$$

8. Shew that if a chord of the parabola $y^2 = 4ax$ touches the parabola $y^2 = 4bx$, the tangents at its extremities meet on the parabola $by^2 = 4a^2x$, and the normals on the curve

$$(4a-b)^3\,y^2 = 4b^2\,(x-2a)^3.$$

9. Two similar co-axial ellipses have the lengths of their corresponding axes in the ratio $\sec\alpha : 1$. From a point on the outer whose eccentric angle is θ tangents are drawn to the inner. Prove that the eccentric angles of the points of contact of these tangents are $\theta+\alpha$ and $\theta-\alpha$.

10. A uniform plank of length $2b$ rests with one end on a rough horizontal plane, touches a smooth fixed cylinder of radius a lying on the plane and makes an angle 2α with the plane, the angle of friction being ϵ. Shew that equilibrium is possible if

$$a\sin\epsilon > b\tan\alpha\cos 2\alpha\sin(2\alpha+\epsilon).$$

11. Shew that the time of quickest descent from a straight line to a circle in the same vertical plane is

$$\sqrt{\frac{2l}{g}}\sec\frac{\theta}{2},$$

where θ is the inclination of the line to the horizontal, and l the shortest distance between it and the circle.

12. A particle is projected from a point in an inclined plane. At the rth impact it strikes the plane at right angles, and at the nth impact it is again back at the point of projection. Prove that

$$e^n - 2e^r + 1 = 0,$$

where e is the coefficient of restitution.

XIII.

1. A transversal cuts the sides of a triangle ABC in P, Q, R, and also cuts three concurrent lines through A, B, and C in P', Q', R'. Prove that

$$PQ' \,.\, QR' \,.\, RP' = -P'Q \,.\, Q'R \,.\, R'P.$$

2. In a parabola if perpendiculars be drawn from the focus to the sides of a circumscribed equilateral triangle, the reciprocal of one of them is equal to the sum of the reciprocals of the other two.

3. Prove that, if $A = a^2 + b^2 + c^2$, $B = bc + ca + ab$, then

$$\begin{vmatrix} B, & A, & B \\ B, & B, & A \\ A, & B, & B \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2.$$

4. Prove that all even numbers which when divided by 7 and 9 give remainders 4 and 3 respectively are of the form $126p - 24$, where p is an integer.

5. Find the number of roots of the equation $(a + bx) \tan x = 1$ lying between any given limits. Shew also that the large roots are approximately given by

$$n\pi + (a + nb\pi)^{-1},$$

where n is any large integer.

6. The right angle C of a right-angled triangle ABC is divided into n equal parts by lines which meet the hypotenuse AB in points P_1, P_2, ... P_{n-1}. Shew that

$$\frac{1}{CP_1} + \frac{1}{CP_2} + \dots + \frac{1}{CP_{n-1}} = \frac{a+b}{2ab}\left(\cot\frac{\pi}{4n} - 1\right),$$

where $BC = a$, and $CA = b$.

7. A straight line is drawn perpendicular to the axis of a parabola. With any point P on it as centre, and any radius, a circle is drawn cutting the parabola in four points. Shew that the sum of the focal distances of the four points is the same for all points P on the line.

8. A parallelogram circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and two of its corners move on $Ax^2+2Hxy+By^2=1$. Shew that the other two move on

$$Bb^4x^2-2Ha^2b^2xy+Aa^4y^2=b^2x^2+a^2y^2-a^2b^2.$$

9. If O be any point on the circumcircle of a triangle ABC, and if OA, OB, OC meet the opposite sides in a, b, c respectively, shew that the straight line on which lie the points of intersection of ab and AB, bc and BC, ca and CA, passes through a point which is independent of the position of O.

10. A regular tetrahedron of height h has a tetrahedron of height xh cut off by a plane parallel to the base. When the remaining frustum is placed on one of its slant faces on a horizontal plane, it is just on the point of falling over. Shew that x is a root of the equation

$$x^3+x^2+x=2.$$

11. A wet open umbrella is held upright with its rim (of radius a) at a height h above the ground, and is rotated about the handle with uniform angular velocity ω. Prove that the drops of water which fly off from the rim will, on reaching the ground, lie on a circle of radius

$$a\left(1+\frac{2\omega^2h}{g}\right)^{\frac{1}{2}}.$$

12. Two equal balls A and B are placed on an inclined plane and start from rest at the same instant. After the lower ball B has passed over a space b it impinges perpendicularly on a plane (coefficient of elasticity e) and after rebounding comes to rest just as it strikes A. Prove that

$$\frac{a}{b}=2e(e+1),$$

where a is the initial distance between the balls.

XIV.

1. The concurrent lines AL, BM, CN through the angular points of a triangle ABC meet the opposite sides in L, M, N respectively and L' is the harmonic conjugate with respect to B and C of the point L. Prove that the points M, N, L' are collinear.

2. P is a point on a hyperbola, PN its ordinate, M a point on the conjugate axis such that $AM = CP$. Prove that if S be a focus,

$$CM : PN = CS : CB,$$

and if P', M' be a corresponding pair of points, $PM' = P'M$.

3. Prove that, n being a positive integer,

$$1^2 \,.\, 2^2 + 2^2 \,.\, 3^2 + \dots + n^2 (n+1)^2 \equiv \frac{n(n+1)(n+2)}{15}(3n^2 + 6n + 1).$$

4. Sum the series

$$\text{(i)} \quad x^n - n(x+y)^n + \frac{n(n-1)}{2\,!}(x+2y)^n - \dots ,$$

$$\text{(ii)} \quad \frac{3}{1\,.\,3} + \frac{7}{2\,.\,4} \,.\, 3 + \frac{11}{3\,.\,5} \,.\, 3^2 + \frac{15}{4\,.\,6} \,.\, 3^3 + \dots ,$$

each to $(n+1)$ terms.

5. If $A + B + C = \pi$, and

$$x = \Sigma \sin A, \quad y = \Sigma \sin^2 A, \quad z = \sin A \sin B \sin C,$$

then
$$4z^2 + 8xz + 2x^2 y = x^4.$$

6. Prove that $\tan 9°$ is a root of the equation

$$x^4 - 4x^3 - 14x^2 - 4x + 1 = 0,$$

and find the other roots.

7. Pairs of circles are drawn having external contact with each other and also with the circles

$$(x-a)^2 + y^2 = r^2, \quad (x - ma)^2 + y^2 = m^2 r^2.$$

Shew that the points in which the pairs of circles touch each other lie on the circle

$$x^2 + y^2 = m(a^2 - r^2).$$

8. Trace the curve

$$6x^2 + 24xy - y^2 + 84x - 12y - 60 = 0.$$

9. Three tangents to a parabola form a triangle PQR and the circle PQR cuts the parabola in the four points A, B, C, D. Prove that, if S be the focus, then

$$\frac{SA \,.\, SB \,.\, SC \,.\, SD}{SP \,.\, SQ \,.\, SR}$$

is constant.

10. A uniform rod of weight $2W$ and length $4a$ is bent to a right angle at its middle point, and is there fastened by a pivot about which it is free to turn. Two smooth rings, of weights w and w', rest, one upon each of the arms of the rod, and are joined by a light string of length l. Shew that the inclination θ of the string to the arm supporting w is given by

$$w(Wa + wl \cos\theta)\sin\theta = w'(Wa + w'l \sin\theta)\cos\theta.$$

11. An engine draws a train along a level line, starting from rest. If the pull of the engine be constant till steam is shut off, and the resistance F be constant throughout the journey, then the greatest rate of working is

$$\frac{2lF^2t}{Ft^2 - 2Ml},$$

where M is the mass of the train, l the length of the journey, and t the time occupied by it.

12. Prove that the maximum range on a horizontal plane of missiles projected from a hill of height h with velocity V is

$$\frac{V^2}{g}\left(1 + \frac{2gh}{V^2}\right)^{\frac{1}{2}}.$$

If h be 300 ft. and V be 1,000 ft. per sec., find over what distance a similar battery on the plane would be under fire, and unable to return it.

XV.

1. Find the locus of a point at which two given portions of the same straight line subtend equal angles.

2. The angle between any two chords of a conic is equal to the angle subtended at the focus by the portion of the directrix intercepted by the diameters which bisect the chords.

3. Shew that the numbers which in any scale of notation are represented by 1, 11, 111, 1111, ... form a recurring series, whose scale of relation is $u_{n+2} - (r+1)\,u_{n+1} + ru_n = 0$, where r is the radix. Shew also that

$$\sqrt[n]{u_{n+1}{}^2 - u_n u_{n+2}} = r.$$

4. If n be any integer, prove that

$$\left\{\frac{n\,(n+1)^3}{8}\right\}^n > (n\,!)^4.$$

5. If each of the legs (supposed equal) of a camera tripod makes an angle α with the ground, their feet forming an equilateral triangle, prove that the angle between any two of the legs is

$$\cos^{-1}\{\tfrac{1}{4}\,(1 - 3\cos 2\alpha)\}.$$

6. Prove that $4\cos^2\dfrac{\pi}{7}$ is a root of the cubic

$$x^3 - 5x^2 + 6x - 1 = 0,$$

and determine the other roots.

7. The tangents at the points Q and R of the parabola $y^2 = 4ax$ intersect at P. The perpendicular from P on QR meets the axis at G, and S is the focus. Prove that the radius of the circle PQR is

$$SP\,.\,PG \div 2a.$$

8. Shew that the line joining the pole of a normal chord of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its pole with regard to

$$\frac{x^2}{a^2 + n^2 b^2} + \frac{y^2}{b^2 + n^2 a^2} = 1$$

touches the ellipse

$$a^2 x^2 + b^2 y^2 = (a^2 + b^2)^2.$$

9. A series of rectangular hyperbolas is drawn, each passing through a given point and touching a given line at a given point. Prove that their centres lie on a circle.

10. Two bars, each of weight W and length $2a$, are freely jointed at a common extremity and rest symmetrically in contact with two pegs at a distance $2c$ apart in the same horizontal line so as to include an angle 2α, their other extremities being connected by a string. Prove that the tension of the string is

$$\tfrac{1}{2} W \sec \alpha \operatorname{cosec}^2 \alpha \left(\frac{c}{a} - \sin^3 \alpha \right).$$

11. Prove that the least velocity with which a particle must be projected from a point on the ground a feet in front of a wall h feet high in order to pass over the wall is

$$\{g\,(h + \sqrt{h^2 + a^2})\}^{\frac{1}{2}}.$$

12. A wedge of mass M and angle α rests on a smooth horizontal plane. A perfectly elastic particle of mass m impinges normally on it with velocity u. Shew that the particle rebounds with velocity

$$\frac{M - m \sin^2 \alpha}{M + m \sin^2 \alpha} u,$$

assuming that no kinetic energy is lost by the impact.

XVI.

1. If in a tetrahedron $ABCD$, the directions of AB, CD be perpendicular, and also those of AC, BD, so also will the directions of BC, AD, and in such a tetrahedron the four perpendiculars from the vertices on the opposite faces and the three shortest distances between pairs of opposite edges will all meet in a point.

2. Given a tangent to an ellipse, its point of contact and the director-circle, construct the ellipse.

3. If $\quad \dfrac{1}{x+a}+\dfrac{1}{y-b}=\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{x+a'}-\dfrac{1}{y-b'}=\dfrac{1}{c}$,

prove that

$$c^2(ab'-a'b)^2=aa'bb'(a-a')(b-b').$$

4. Prove that the product of the first n convergents of the continued fraction $\dfrac{4}{3+}\,\dfrac{4}{3+}\,\dfrac{4}{3+}\dots$ is

$$\frac{5\,.\,4^n}{4^{n+1}+(-1)^n}.$$

5. If α, β, γ, δ are solutions of the equation

$$\cos 2x+a\cos x+b\sin x+c=0,$$

no two of which differ by a multiple of π, prove that $\alpha+\beta+\gamma+\delta$ is a multiple of 2π.

6. If $\alpha=\dfrac{\pi}{13}$, prove that

$$\tan\alpha\tan 2\alpha\tan 3\alpha\tan 4\alpha\tan 5\alpha\tan 6\alpha=\sqrt{13}.$$

7. If the product of the lengths of the tangents drawn from a point P to the parabola $y^2=4ax$ is equal to the product of the focal distance of P and the latus-rectum, prove that the locus of P is the parabola

$$y^2=4a(x+a).$$

8. A chord of an ellipse subtends a right angle at the point on the ellipse of eccentric angle α. Shew that it passes through the fixed point

$$\left(a\frac{a^2-b^2}{a^2+b^2}\cos\alpha,\quad b\frac{b^2-a^2}{b^2+a^2}\sin\alpha\right).$$

9. Conics are drawn having the origin for a common focus, and touching the conic $\frac{l}{r}=1+e\cos\theta$ at the point $\theta=\alpha$. Prove that the least eccentricity of any such conic is

$$e\sin\alpha\,(1+2e\cos\alpha+e^2)^{-\frac{1}{2}},$$

and that the latus-rectum of this conic is

$$2l\,(1+e\cos\alpha)\,(1+2e\cos\alpha+e^2)^{-1}.$$

10. Two rings of weights w and w' are attached to a string which supports a weight, and the rings are free to slide on two smooth straight wires inclined to the horizon at angles α and β. Prove that if each part of the string makes an angle θ with the adjacent wire, then

$$w\left[\cot\beta-\tan(\theta-\beta)\right]=w'\left[\cot\alpha-\tan(\theta-\alpha)\right].$$

11. Two equal spheres, centres A and B, lie in contact on a smooth table and A is struck directly by a third sphere, centre C, moving with velocity V in a direction making an acute angle θ with AB. Shew that after impact A moves in a direction making an angle

$$\cot^{-1}(\cot\theta+2\tan\theta)$$

with CA produced.

12. Two particles are projected simultaneously from a point A so as to pass through another point B, the velocity of projection in each case being V. If α, α' are the angles of projection, prove that the particles will pass through B at times separated by the interval

$$\frac{2V}{g}\frac{\sin\frac{1}{2}(\alpha-\alpha')}{\cos\frac{1}{2}(\alpha+\alpha')}.$$

XVII.

1. If O is any point on the axis of a harmonic range $(ABCD)$, and M, N the middle points of AB, CD, prove that

(i) $OA \,.\, OB + OC \,.\, OD = 2OM \,.\, ON$,

(ii) $OA \,.\, BC + OB \,.\, AD = OC \,.\, BD + OD \,.\, AC$.

2. PQ is any diameter of an elliptic section of a cone whose vertex is V. Prove that $VP + VQ$ is constant.

3. Eliminate x, y, z from the equations

$$\begin{aligned} x - 2y + z &= a, \\ x^2 - 2y^2 + z^2 &= b^2, \qquad \frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 0. \\ x^3 - 2y^3 + z^3 &= c^3. \end{aligned}$$

4. Prove that the number of permutations 6 together of 9 things, 5 of which are alike and the rest all different, is 1044.

5. If a, b, c, d be the lengths of the sides of a quadrilateral of area Δ inscribed in a circle of radius R, prove that

$$16R^2\Delta^2 = (bc + ad)(ca + bd)(ab + cd).$$

6. Shew that

$$\sec^2\frac{\pi}{14} + \sec^2\frac{3\pi}{14} + \sec^2\frac{5\pi}{14} = 8.$$

7. O is the centre of a circle of radius a. P, Q, R are any three points, Δ' the area of the triangle formed by their polars with regard to the circle, Δ the area of the triangle PQR and Δ_1, Δ_2, Δ_3 the areas of the triangles QOR, ROP, POQ. Prove that

$$4\Delta'\Delta_1\Delta_2\Delta_3 = a^4\Delta^2.$$

8. If the normals to the parabola $y^2 = 4ax$ which meet at the point (h, k) make angles ϕ_1, ϕ_2, ϕ_3 with the positive direction of the axis of x, prove that

$$\phi_1 + \phi_2 + \phi_3 = \tan^{-1}\frac{k}{a - h}.$$

9. Prove that the middle points of the diagonals of the quadrilateral formed by the four straight lines

$$\alpha = 0,\ \ \beta = 0,\ \ \gamma = 0,\ \ \frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0$$

lie on the straight line

$$al\,(b\beta + c\gamma - a\alpha) + bm\,(c\gamma + a\alpha - b\beta) + cn\,(a\alpha + b\beta - c\gamma) = 0.$$

10. Three uniform beams of combined weight W and lengths a, $2a$, a are connected by hinges at B, C and rest on a smooth sphere of radius na, so that A, the middle point of BC, and D are in contact with the sphere. Shew that the pressure on the middle point of BC is

$$\frac{1}{4}\frac{(3n^2+1)(n^2+3)}{(n^2+1)^2}W.$$

11. AB and BC are two uniform rods of the same material, freely jointed at B. The ends A and C are fixed in the same vertical. Prove (graphically or otherwise) that the stress at the joint is

$$\tfrac{1}{2}W \cdot \frac{BD}{AC},$$

where BD is the bisector of the angle ABC, and W the weight of the rods.

12. A man h feet high fires a pistol from the level of his head, the ball from which issues with a velocity of V feet per second, just clears a wall a feet high, and strikes the ground as far beyond the wall as the man is from it. Shew that the pistol was held at an elevation

$$\sin^{-1}\left\{\frac{4a-3h}{2V}\sqrt{\frac{g}{2a-h}}\right\}.$$

XVIII.

1. Through the intersection of the diagonals of a quadrilateral lines are drawn respectively parallel to the four sides and intersecting the sides opposite to those to which they are drawn parallel. Prove that the four points of intersection lie on a straight line.

2. The axes of an ellipse are $2a$, $2b$. With the centre O of the ellipse as centre and with radii a, b, $a+b$ circles are described and a radius vector meets them in P, Q and R respectively. If a parallel to the minor axis through P meets a parallel to the major axis through Q in S, then S is a point on the ellipse, and SR is the normal at S.

3. Shew that the sum of all the homogeneous products of a, b, c of all dimensions from 0 to n is

$$\Sigma \frac{a^{n+3}}{(a-b)(a-c)(a-1)} - \frac{1}{(a-1)(b-1)(c-1)}.$$

4. Prove that the recurring series whose scale is

$$u_n - pu_{n-1} + q^2 u_{n-2} = 0,$$

where p and q are real, will be convergent if p lies between $2q$ and q^2+1.

5. If the equations

$$\tan x \tan (y-z) = a, \;\; \tan y \tan (z-x) = b, \;\; \tan z \tan (x-y) = c$$

are consistent, shew that

$$a+b+c+abc=0.$$

6. From G, the centre of gravity of a triangle ABC, perpendiculars GP, GQ, GR are let fall on the sides. Shew that

(i) the area of the triangle PQR is

$$\tfrac{1}{18}(a^2+b^2+c^2)\sin A \sin B \sin C,$$

(ii) the radius of the circle PQR is

$$\tfrac{4}{3} AD \,.\, BE \,.\, CF/(a^2+b^2+c^2),$$

(iii) the sum of the areas of the circles PGQ, QGR, RGP is

$$\frac{\pi}{12}(a^2+b^2+c^2).$$

7. Shew that the common tangents to the two circles

$$x^2 + y^2 - 2x - 2y + 1 = 0, \;\; x^2 + y^2 - 8x - 8y + 28 = 0$$

are the lines

$$x = 2, \;\; y = 2 \text{ and } 8x + (-9 \pm \sqrt{17})\, y - 2 \pm 2\sqrt{17} = 0.$$

8. Shew that the inclinations θ to the axis of x of the tangents drawn from the point (p, q) to the conic $ax^2 + by^2 = 1$ are determined by the equation

$$(ap^2 + bq^2 - 1)(a + b \tan^2\theta) = (ap + bq \tan\theta)^2,$$

and also that the square of the length of the tangent whose inclination θ is thus determined is

$$\frac{1 + \tan^2\theta}{a + b\tan^2\theta}(ap^2 + bq^2 - 1).$$

9. Trace the curve

$$3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0,$$

and find the equations of its directrices.

10. A rod of length $2a$ inclined at an angle θ to the horizontal rests tangentially against a fixed rough cylinder with its axis horizontal, being held in position by a horizontal string attached to its highest point and perpendicular to the axis of the cylinder. Find the friction at the point of contact, and prove that the part of the rod above the point of contact must lie between the limits

$$a\cos\theta\,(\cos\theta \pm \mu\sin\theta),$$

where μ is the coefficient of friction.

11. A mass rests on an inclined plane supported by a string which passes over a smooth pulley at the summit of the plane, and carries at its other extremity a mass M which hangs vertically. If the mass M be removed and replaced by a pulley of mass p, over which passes a string with masses m and m' at its extremities, shew that the mass on the plane will still remain at rest provided

$$(m + m')(M - p) = 4mm'.$$

12. A regular hexagon stands with one side on the ground and a particle is projected so as just to graze the four upper corners. Shew that the velocity of the particle on reaching the ground is to its least velocity as $\sqrt{31} : \sqrt{3}$.

XIX.

1. Shew that the four orthocentres of the four triangles formed by the sides of a quadrilateral are collinear.

2. The normal at a point P of an ellipse meets the axes in G, g. The foci are S and S', and the circle gGS meets SP in Q. Prove that $PQ = S'P$, and that the diameter conjugate to CP bisects SQ.

3. If
$$a(b-c)x + b(c-a)y + c(a-b)z = 0,$$
$$(b-c)x^2 + (c-a)y^2 + (a-b)z^2 = 0,$$
then will $x = y = z$, or
$$\frac{x}{-bc+ca+ab} = \frac{y}{bc-ca+ab} = \frac{z}{bc+ca-ab}.$$

4. Solve graphically the following equations:

(i) $10x^3 + 3 = 15x$,

(ii) $x \log_e x = 1$.

5. If p, q, r be the lengths of the bisectors of the angles of a triangle produced to meet the circumcircle and u, v, w the lengths of the perpendiculars of the triangle produced to meet the same circle, prove that
$$p^2(v-w) + q^2(w-u) + r^2(u-v) = 0.$$

6. If the equation
$$\cot(\theta - \alpha_1) + \cot(\theta - \alpha_2) + \cot(\theta - \alpha_3) = 0$$
has solutions θ_1, θ_2, θ_3 not differing by multiples of two right angles, prove that
$$\theta_1 + \theta_2 + \theta_3 - \alpha_1 - \alpha_2 - \alpha_3$$
is an odd multiple of a right angle.

7. Tangents are drawn from a given point (h, k) to a system of confocal and co-axial parabolas. Shew that the normals at the points of contact intersect on the line
$$hx + ky + h^2 + k^2 = 0.$$

8. If the circle
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

passes through the extremities of three semi-diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

prove that the circle

$$x^2 + y^2 + 2\frac{fb}{a}x - 2\frac{ga}{b}y - (a^2 + b^2 + c) = 0$$

passes through the extremities of the three conjugate semi-diameters.

9. Prove that the area of the ellipse

$$ax^2 + by^2 + px + qy + r = 0$$

is
$$\frac{\pi}{\sqrt{ab}}\left(\frac{p^2}{4a} + \frac{q^2}{4b} - r\right).$$

10. A triangle of rods smoothly jointed at A, B, C is hung up by the corner A. Prove that the action at the joint C is inclined to the horizontal at an angle

$$\tan^{-1}\frac{(2W_1 + W_3)\cot\phi - W_2\cot\theta}{2W_1 + W_2 + W_3},$$

where W_1, W_2, W_3 are the weights of BC, CA, AB and θ, ϕ are the inclinations of AB, AC to the vertical.

11. Two rods AB, AC, each of weight W' and length $2a$, are rigidly connected so that they are at right angles and can turn freely in a vertical plane about a pivot at A. Two small rough rings, each of weight W, are placed one on each rod and are connected by a light string of length $2l$, passing over a smooth pulley at A. If μ be the coefficient of friction, prove that the distances of the rings from A must lie between

$$l \pm \mu\left(l + \frac{W'}{W}a\right),$$

provided
$$\mu < \frac{Wl}{Wl + W'a}.$$

12. A rocket fired vertically upwards bursts at its highest point h feet above the ground. If each fragment starts with the same velocity U, prove that all the fragments on reaching the ground lie within a circle of radius

$$\frac{U}{g}\sqrt{U^2 + 2gh}.$$

XX.

1. A straight line PQ is drawn parallel to AB to meet the circumcircle of the triangle ABC in P and Q. Shew that the pedal lines of P and Q intersect on the perpendicular from C on AB.

2. The tangent at a fixed point P of an ellipse whose foci are S and S' meets a pair of conjugate diameters in T and T'. Shew that the locus of the other intersection of the circles SPT, $S'PT'$ is a circle.

3. If n be a positive integer, shew that

$$1-\frac{n(n+1)}{1^2}+\frac{n(n-1)(n+1)(n+2)}{1^2 . 2^2} - \frac{n(n-1)(n-2)(n+1)(n+2)(n+3)}{1^2 . 2^2 . 3^2} + \ldots + (-1)^n\frac{(2n)!}{(n!)^2} \equiv (-1)^n.$$

4. Shew that the fourth power of the infinite continued fraction

$$1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\ldots$$

is the infinite continued fraction

$$7-\frac{1}{7-}\,\frac{1}{7-}\ldots.$$

5. Prove that if a, b, c, x, y, z are rational, and

$$a+b+c=0, \quad x+y+z=0,$$

the area of the triangle whose sides are

$$\sqrt{a^2+x^2}, \quad \sqrt{b^2+y^2}, \quad \sqrt{c^2+z^2},$$

is rational and equal to

$$\tfrac{1}{2}\{-(bcx^2+cay^2+abz^2)\}^{\frac{1}{2}}.$$

6. If the sine of a small angle be taken as nearly equal to its circular measure, find (without using Tables) to how many places of decimals this approximation will be correct in the case of an angle of 5°.

7. The polars of any three points with respect to the parabola $y^2 = 4ax$ form a triangle of area Δ_1: the tangents parallel to them form a triangle of area Δ_2 and Δ is the area of the triangle ABC. Shew that

$$4\Delta_1\Delta_2 = \Delta^2.$$

8. Prove that the length of the normal chord of the ellipse of axes $2a$, $2b$ which makes equal angles with the axes is

$$4\sqrt{2}\,a^2b^2(a^2+b^2)^{-\frac{3}{2}}.$$

9. Find the focus and directrix of the parabola

$$9x^2 + 30xy + 25y^2 - 206x + 246y + 393 = 0.$$

10. Three equal uniform rods of length l and weight w are smoothly jointed together to form a triangle ABC. This triangle is hung up by the joint A, and a weight W is attached to B and C by two strings each of length $\dfrac{l}{\sqrt{2}}$. If the system hangs under gravity, shew that the thrust along BC is equal to

$$\frac{1}{\sqrt{3}}\left\{w + \frac{W}{2}(1+\sqrt{3})\right\}.$$

11. A wedge of mass m_1 and angle α lies on a horizontal table, and a second wedge of the same angle and mass m_2 is placed upon it so that the upper face is horizontal. Upon this face is placed a particle of mass m_3. Shew that, in the ensuing motion, the total weight will exceed the pressure on the table by

$$\frac{(m_1+m_2)(m_2+m_3)^2 g\sin^2\alpha}{(m_1+m_2)(m_2+m_3)\sin^2\alpha + m_1m_2\cos^2\alpha}.$$

12. An elastic particle is projected with velocity V from a point on the ground and strikes a smooth vertical wall with its foot at a distance a from the point of projection. Prove that after rebounding from the wall, the particle cannot strike the ground at a point further from the wall than the point of projection unless

$$V^2 > \frac{1+e}{e}\,.\,ga.$$

XXI.

1. The sides BC, CA, AB of a triangle meet any straight line in the points D, E, F. Shew that a point P can be found in the line DEF, such that the areas PAD, PBE, PCF are equal.

2. The continued products of the focal vectors of any three points on a parabola, and of those of the poles of the chords joining the three points, are equal.

3. If the system of equations

$$x + y + z = 0, \quad ax^2 + by^2 + cz^2 = 0, \quad ax^4 + by^4 + cz^4 = 0$$

admit of a solution other than $x = y = z = 0$, then

$$(b + c)(c + a)(a + b)\{(b + c)(c + a)(a + b) - 8abc\} = 0.$$

4. A plant produces p seeds at the end of the second year of its life, p more at the beginning of the third year, and dies at the end of its third year. Prove that if all the seeds come to maturity the number produced from one seed in the nth year from planting that seed is

$$2^{-n} q^{-1} \{(p + q)^n - (p - q)^n\},$$

where $q^2 = p^2 + 4p$.

5. If t_1, t_2, t_3 be the tangents from the centre of the nine-point circle of a triangle to the escribed circles, prove that

$$\frac{t_1^2}{r_1} + \frac{t_2^2}{r_2} + \frac{t_3^2}{r_3} = \frac{R + 12r}{16 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}.$$

6. Prove that if s be the sum of the four values of θ less than 2π which satisfy the equation

$$a \cos 2(\theta - \alpha) + b \cos(\theta - \beta) + c = 0,$$

then $s = 2n\pi + 4\alpha$, where n is some integer.

7. If the origin be at one limiting point of the system of coaxial circles of which $x^2 + y^2 + 2gx + 2fy + c = 0$ is a member, shew that the system of orthogonal circles is

$$(x^2 + y^2)(g + \mu f) + c(x + \mu y) = 0.$$

8. Prove that if the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \alpha x^2 + \beta y^2 + 2\gamma xy = 1$$

intersect at right angles, then

$$\frac{\alpha a^2 - 1}{\alpha a^2 b^2} = \frac{2}{a^2 + b^2} = \frac{\beta b^2 - 1}{\beta a^2 b^2}.$$

9. Shew that one focus of the conic

$$x^2 + y^2 + 2hxy + 2g(x+y) + \frac{g^2}{h} = 0$$

is the origin, and that the other is the point

$$x = y = -\frac{2g}{1+h}.$$

10. Three equal uniform heavy rods AB, BC, CD are jointed together at B and C, and the whole system is then suspended from two points A and D in the same horizontal line. If α is the inclination of AB or CD to the vertical, and if θ, ϕ be the inclinations to the vertical of the stresses at A and B, prove that

$$\tan\phi = 2\tan\alpha = 3\tan\theta.$$

11. A hexagonal framework is made of six light smoothly-jointed wires, and has six equal particles attached to its angular points. One of its sides is maintained in a fixed horizontal position and the other sides are held in the same straight line with it and then released. Shew that when the framework is a regular hexagon the velocity of separation of the two horizontal sides will be

$$\left\{\frac{3\sqrt{3}}{2}ag\right\}^{\frac{1}{2}},$$

where a is the length of a side.

12. A ball A of mass nm impinges obliquely on another B, of mass m, at rest. Shew that there will in general be two positions in which B can be placed, so that the direction of A's motion may be turned through a given angle α, if

$$n < \frac{1+e}{2}\operatorname{cosec}\alpha - \frac{1-e}{2}.$$

If, however, there be only one position for B, and if $e = 1$, $\alpha = \frac{\pi}{3}$, shew that $n = \frac{2}{\sqrt{3}}$ and the velocities of A before and after impact, and that of B, will be as

$$\cos\frac{\pi}{12} : \sin\frac{\pi}{12} : 1.$$

XXII.

1. The tangents TP, TP' to a circle are bisected in M, M' and the lines joining P, P' to any point Q on the circle cut MM' in R, R'. Shew that T, R, R', Q lie on a circle.

2. From a point T on the director circle of an ellipse are drawn two lines, tangents to the conic, and meeting the circle again in P, Q. Shew that PQ is conjugate, with respect to the conic, to the line connecting T with the centre.

3. Prove that the coefficient of x^n in the expansion of

$$\frac{1}{(1-ax)^2(1-bx)}$$

is

$$\frac{(n+1)\,a^{n+2}-(n+2)\,a^{n+1}b+b^{n+2}}{(a-b)^2},$$

x being numerically less than the lesser of the two quantities $\frac{1}{a}$, $\frac{1}{b}$.

4. Shew that if m be an integer prime to 30, then m^4-1 is divisible by 240; and that the necessary and sufficient condition that m^2-1 should be divisible by 24 and m^2+1 by 10 simultaneously, is that m should be of one of the forms $30k \pm 7$ or $30k \pm 13$, where k is any integer.

5. If ρ_1, ρ_2, ρ_3 be the distances of any point in the plane of an equilateral triangle of side a from the angular points, prove that

$$\Sigma\rho_2{}^2\rho_3{}^2-\Sigma\rho_1{}^4+a^2\Sigma\rho_1{}^2-a^4=0.$$

6. Shew that if $\cos(\alpha+\theta)=\cos\alpha\cos\phi-\cos\beta\sin\alpha\sin\phi$, where θ and ϕ are small, then θ is very approximately equal to

$$\phi\cos\beta+\tfrac{1}{2}\phi^2\cot\alpha\sin^2\beta.$$

7. Shew that the lines $ax^2-2hxy+by^2=0$ will form an equilateral triangle with $x\cos\alpha+y\sin\alpha=p$ if

$$\frac{a}{1-2\cos 2\alpha}=\frac{h}{2\sin 2\alpha}=\frac{b}{1+2\cos 2\alpha}.$$

8. Shew that the limiting points of the system

$$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + c') = 0$$

subtend a right angle at the origin if

$$\frac{c}{g^2} + \frac{c'}{f^2} = 2.$$

9. Shew that the sum of the squares of the normals from (ξ, η) to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$2\left\{a^2 + b^2 + \frac{a^2 - 2b^2}{a^2 - b^2} \cdot \xi^2 + \frac{b^2 - 2a^2}{b^2 - a^2} \cdot \eta^2\right\}.$$

10. Two unequal rods BC, CD of weights w_1, w_2 are freely jointed at C, and are suspended from two fixed points A, E not in the same vertical line by short strings AB, ED. When the system is in equilibrium the strings make angles α, β with the vertical. Shew that the stress at C makes an angle θ with the vertical given by

$$(w_1 + w_2) \cot\theta = w_2 \cot\alpha - w_1 \cot\beta.$$

11. A particle is projected from any point, and at the same time an equal particle is let fall from a point on the directrix of its path. If the particles meet and coalesce, shew that the tangent to the new path at the point of union is at right angles to the original direction of projection, and the height of the directrix above the original point of projection is three-quarters of the height of the directrix of the first path.

12. A smooth sphere of mass m' is suspended from a fixed point by an inextensible string. Another smooth sphere of mass m falling vertically impinges on the first with a velocity u. Prove that the initial velocity of the first sphere is

$$\frac{mu\cos\theta\sin\theta(1+e)}{m' + m\sin^2\theta},$$

where θ is the inclination of the line joining the centres of the spheres to the vertical at the moment of impact, and e is the coefficient of restitution.

XXIII.

1. Through a given point within a parallelogram draw a straight line to divide the parallelogram into two parts as unequal as possible.

2. The tangents at the ends of a focal chord of a conic intersect at T and the normals at N. Shew that TN passes through the other focus.

3. The four quantities a, b, m, n being supposed positive, shew that unless a and b are equal

$$(ma+nb)^{m+n} > (m+n)^{m+n} a^m b^n.$$

4. If
$$x = \frac{a}{a'+} \frac{b}{b'+} \frac{a}{a'+} \frac{b}{b'+} \dots,$$
$$y = \frac{b}{b'+} \frac{a}{a'+} \frac{b}{b'+} \frac{a}{a'+} \dots,$$
then will
$$a'x - b'y = a - b.$$

5. If α and β are values of θ satisfying the equation

$$a \tan\theta + b \sec\theta = c$$

and $\alpha - \beta$ is not a multiple of π, shew that

$$\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}, \quad \tan(\alpha-\beta) = \frac{\pm 2b(a^2+c^2-b^2)^{\frac{1}{2}}}{2b^2-a^2-c^2}.$$

6. Shew that when $\alpha = \beta$, the limit of

$$\frac{\alpha\sin\beta - \beta\sin\alpha}{\alpha\cos\beta - \beta\cos\alpha}$$

is $\tan(\alpha - \tan^{-1}\alpha)$.

7. Shew that the equation to the straight lines joining the origin to the points of intersection of the conics

$$u_0 + u_1 + u_2 = 0, \quad v_0 + v_1 + v_2 = 0,$$

(where u_n, v_n are homogeneous functions of x and y of order n) is

$$(u_0v_1 - u_1v_0)(u_1v_2 - u_2v_1) = (u_0v_2 - u_2v_0)^2.$$

Hence find the condition that the conics

$$ax^2 + by^2 = gx, \quad a'x^2 + b'y^2 = f'y$$

may have contact of the second order.

8. Prove that two parabolas can be drawn through the feet of the normals from (h, k) to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and that their latera-recta are

$$\frac{2a^2b^2(ah \pm bk)}{(a^2 - b^2)(a^2 + b^2)^{\frac{3}{2}}}.$$

9. P and Q are points on the equilateral hyperbola $xy = k^2$, such that the osculating circle at P passes through Q. Shew that the locus of the pole of PQ is

$$(x^2 + y^2)^2 = 4k^2xy.$$

10. Two spheres are supported by strings attached to a fixed point and rest against one another. If P, Q are their weights, a, b the distances of their centres from the given point, ω the angle between the strings, prove that the tensions of the strings are

$$\frac{P(P + Q)a}{(P^2a^2 + 2PQab\cos\omega + Q^2b^2)^{\frac{1}{2}}},$$

and
$$\frac{Q(Q + P)b}{(P^2a^2 + 2PQab\cos\omega + Q^2b^2)^{\frac{1}{2}}}.$$

11. Heavy beads, n in number, are attached to a string at regular distances a apart and lie, heaped together, on a table. One is raised to a height just less than a, and from that position is projected vertically upwards with velocity V. Shew that, if

$$V^2 = \tfrac{1}{3}ga(n - 2)(2n^2 + n + 3),$$

the nth bead will just not rise from the table.

12. A heavy particle projected with velocity u strikes at an angle of 45° an inclined plane of angle β, which passes through the point of projection. Shew that the vertical height of the point struck above the point of projection is

$$\frac{u^2}{g}\frac{1 + \cot\beta}{2 + 2\cot\beta + \cot^2\beta}.$$

XXIV.

1. A triangle is given in species and one vertex is fixed while the others lie one on each of two given circles. Construct the triangle.

2. The focal radii SP and $S'P$ of a point P on a hyperbola meet an asymptote in Q and R. Prove that the perimeter of the triangle PQR is constant.

3. A pure recurring decimal has N figures in its period, of which m are equal to a, n equal to b, etc. Prove that the sum of all the different decimals obtained by interchanging the figures in all possible ways is

$$\frac{1}{9}\frac{(N-1)!}{m!\,n!\,p!\ldots}(ma+nb+pc+\ldots).$$

4. If a and b are primes, shew that $a^{b-1}+b^{a-1}-1$ is divisible by ab.

5. If the cosines of the angles of a triangle are the roots of the equation

$$x^3+px^2+qx+r=0,$$

prove that the angle θ between the lines joining the orthocentre to the centres of the inscribed and circumscribed circles is given by

$$\cos\theta=\frac{1+p+2q+8r}{2\sqrt{(1+8r)(1+p+q+2r)}}.$$

6. If n is an odd integer, prove that

$$\sin\frac{\pi}{4n}\sin\frac{3\pi}{4n}\sin\frac{5\pi}{4n}\ldots\sin\frac{(2n-1)\pi}{4n}=2^{-n}\sqrt{2}.$$

7. Sum the series

(i) $\cot^{-1}(2\,.\,1^2)+\cot^{-1}(2\,.\,2^2)+\cot^{-1}(2\,.\,3^2)+\ldots$ to n terms

(ii) $2\cos\alpha\sin\alpha-\dfrac{4\cos^2\alpha\sin 2\alpha}{2\,!}+\dfrac{8\cos^3\alpha\sin 3\alpha}{3\,!}-\ldots$ to infinity.

8. If (x_1y_1), (x_2y_2), (x_3y_3) are the vertices of a triangle self-conjugate to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, shew that the area of the triangle is

$$\frac{1}{2a^2b^2}(x_2y_3 \sim x_3y_2)(x_3y_1 \sim x_1y_3)(x_1y_2 \sim x_2y_1).$$

9. Shew that the envelope of the chord of the conic

$$ax^2+2hxy+by^2+c=0,$$

the tangents at whose extremities cut at right angles, is the conic

$$(a^2+h^2)x^2+(b^2+h^2)y^2+2(a+b)hxy=\frac{(h^2-ab)c}{a+b}.$$

10. Two equal uniform rods AB, BC are freely jointed at B, and are suspended from a peg at A, the rods being maintained at right angles to one another by a weightless string AC fastened to the peg at A. Find the tension of the string, and prove that the stress across the joint B is equal to $\frac{1}{2}W$, where W is the weight of each rod.

11. A metal pot in the form of a right circular cylinder has a hollow hemispherical lid of material whose density is n times that of the rest of the pot, the lid being closed by a circular disc of the same material as the rest of the pot. Prove that the pot will stand upright on a horizontal table with the lid open at any angle, provided the ratio of the height of the cylinder to its radius is greater than $\sqrt{5n^2+4n+1}-1:2$.

12. A wire ABC in the form of an equilateral triangle is fixed on a horizontal table. A particle is projected from a point in BC in a direction parallel to BA. If the point of projection divides BC in the ratio $2e:3e-1$, prove that the particle will return to it after impinging on AC and AB.

XXV.

1. On the sides BC, CA, AB of a given triangle are taken points P, Q, R such that the triangle PQR is of given species. Prove that the locus of the circumcentre of PQR is a straight line.

2. Shew how to cut from a given right circular cone a parabolic section of given latus rectum.

3. If h_n be the sum of all the homogeneous products of n dimensions wherein no letter is to appear raised to so high a power as m, prove that

$$h_n = H_n - (\Sigma a_1^m) H_{n-m} + (\Sigma a_1^m a_2^m) H_{n-2m} - (\Sigma a_1^m a_2^m a_3^m) H_{n-3m} + \dots,$$

where H_n is the sum of all the homogeneous products of n dimensions.

4. If w_n be the nth convergent to the continued fraction

$$\frac{1}{p_1 +} \frac{1}{p_2 +} \frac{1}{p_3 +} \dots \frac{1}{p_n +} \dots,$$

shew that

$$\frac{(w_{n+1} - w_n)(w_{n-1} - w_{n-2})}{(w_{n+1} - w_{n-1})(w_n - w_{n-2})} + \frac{1}{p_n p_{n+1}} = 0.$$

5. If 78 and 50 be the lengths of the diagonals of a quadrilateral inscribed in a circle of radius 65 and $\sin^{-1}\frac{3}{5}$ the angle between them, shew that the sides of the quadrilateral are $11\sqrt{26}$, $5\sqrt{26}$, $5\sqrt{26}$, $19\sqrt{26}$.

6. Sum the series

$$\frac{b}{c}\sin A + \frac{1}{2}\frac{b^2}{c^2}\sin 2A + \frac{1}{3}\frac{b^3}{c^3}\sin 3A + \dots,$$

where b, c are the two sides of the triangle ABC containing the angle A, and $b < c$.

7. Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$, and the other to $x^2 = 4by$. Shew that the locus of their point of intersection is the curve

$$(x^2 + y^2)(ax + by) + (bx - ay)^2 = 0.$$

8. From any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ three normals other than the one at the point are drawn. Shew that the centre of the circle through their feet lies on the conic

$$a^6x^2 + b^6y^2 = \tfrac{1}{4}a^4b^4.$$

9. ABC is a triangle and conics are drawn touching AB at B and AC at C. Tangents are drawn to these conics parallel to a given straight line. Shew that the locus of their points of contact is a conic circumscribing the triangle ABC.

10. A square lamina whose plane is vertical rests with the ends of one side against a rough vertical wall and a rough horizontal ground. If the coefficients of friction for the ground and the wall are μ and μ' respectively, prove that, when the lamina is on the point of motion, the inclination of the side in question to the horizontal is

$$\tan^{-1}\left(\frac{1 - \mu\mu'}{1 + 2\mu + \mu\mu'}\right).$$

11. A particle of mass m slides down a smooth inclined plane of mass M and angle α. The wedge can slide on a smooth horizontal plane. If h be the initial height of the particle above the plane, shew that it will reach the horizontal plane in a time t given by

$$t^2 = \frac{2h}{g}\left(1 + \frac{M}{M+m}\cot^2\alpha\right),$$

and that in this time the wedge will have moved a horizontal distance

$$\frac{mh}{M+m}\cot\alpha.$$

12. A ball is projected from a point in a smooth plane inclined at an angle α to the horizon in a direction making an angle β with the plane, and in the vertical plane through the plane's normal. If the coefficient of elasticity be e, prove that the condition that the ball should return to the point of projection is that

$$\log\{1 - (1-e)\cot\alpha\cot\beta\}/\log e$$

should be a positive integer.

XXVI.

1. Shew that if each of two pairs of opposite vertices of a quadrilateral is conjugate with regard to a circle, the third pair is also, and that the circle is one of a coaxal system of which the line of collinearity of the middle points of the diagonals is the radical axis.

2. Tangents PQ, $P'Q$ are drawn from the extremities of any diameter PP' of an ellipse to a given concentric circle. Shew that the locus of Q is a conic section and determine under what circumstances this conic section will coincide with the original ellipse.

3. Prove that the value of the determinant

$$\begin{vmatrix} 1+x^2, & x, & 0, & 0 & \ldots \\ x, & 1+x^2, & x, & 0 & \ldots \\ 0, & x, & 1+x^2, & x & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

of the mth order is $1+x^2+x^4+\ldots+x^{2m}$.

4. If $$s_r = \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \ldots + \frac{1}{n^r},$$

prove that

$$\log_e(n+1) = s_1 - \tfrac{1}{2}s_2 + \tfrac{1}{3}s_3 - \ldots \text{ ad inf.}$$

5. If $A'B'C'$ are the feet of the perpendiculars of the triangle ABC; A_1, A_2 the projections of A' on AB, AC; B_1, B_2 those of B' on BC, BA; C_1, C_2 those of C' on CA, CB: shew that

$$a^2 \,.\, \Delta_1 + b^2 \,.\, \Delta_2 + c^2 \,.\, \Delta_3 = \frac{\Delta^3}{R^2},$$

where Δ, Δ_1, Δ_2, Δ_3 are the areas of the triangles ABC, $A'A_1A_2$, $B'B_1B_2$, $C'C_1C_2$ respectively.

6. If $\alpha = \dfrac{2\pi}{13}$ and $\beta = \dfrac{\pi}{9}$, shew that

$$(\cos\alpha + \cos 5\alpha)(\cos 2\alpha + \cos 3\alpha)(\cos 4\alpha + \cos 6\alpha)$$
$$= -\tfrac{1}{8} = -2\cos\beta\cos 2\beta\cos 3\beta\cos 4\beta.$$

7. Shew that

$$\frac{\sin\theta}{1-2ax+a^2x^2\sec^2\theta} = \sum_0^\infty a^n x^n \sec^n\theta \sin(n+1)\theta.$$

8. A triangle is inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and has its centre of gravity at the centre of the ellipse. Shew that the locus of the circumcentre is

$$a^2x^2+b^2y^2=\{\tfrac{1}{4}(a^2-b^2)\}^2.$$

9. Tangents are drawn from a point O to an ellipse so as to intercept a fixed length on the tangent at a given point P. Shew that the locus of O is a conic which has four-point contact with the ellipse at the other extremity P' of the diameter through P.

10. Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the whole weight.

11. If U and u be the velocities of two particles whose weights are W and w, and if ρ be their relative velocity, V the velocity of their centre of inertia, and θ the angle between V and ρ, prove that

$$U^2=V^2+\left(\frac{w}{W+w}\rho\right)^2+\frac{2w}{W+w}V\rho\cos\theta,$$

$$u^2=V^2+\left(\frac{W}{W+w}\rho\right)^2-\frac{2W}{W+w}V\rho\cos\theta.$$

12. A particle is projected along the inside of a smooth sphere of radius a from its lowest point, so that after leaving the sphere it describes a free path passing through the lowest point. Prove that the velocity of projection is $\sqrt{\tfrac{7}{2}ag}$.

XXVII.

1. Two circles intersect orthogonally at a point P, and O is any point on any circle which touches the two former circles at Q and Q'. Shew that the angle of intersection of the circles OPQ, OPQ' is half a right angle.

2. The centres of two circles of radii a and b are distant d apart. Prove that the latus-rectum of the parabola which has double contact with both circles is $(a^2 \sim b^2)/d$.

3. If x is a prime number greater than 3, and $x^2 + y^2 = z^2$, shew that y is of the form $12m$, and that $2(y + x + 1)$ is a square.

4. Prove that the coefficient of x^n in the expansion of

$$\frac{1 + x}{(1 + x^2)(1 - x)^2}$$

is

$$\tfrac{1}{2}[2n + 3 - (-1)^p],$$

where p is either $\frac{1}{2}n$ or $\frac{1}{2}(n + 1)$, according as n is even or odd.

5. Prove the identity

$$\frac{(1 - \cos 14\theta)(1 - \cos\theta)}{(1 - \cos 2\theta)(1 - \cos 7\theta)} = (2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1)^2.$$

6. Shew that for an angle of 10° the value of θ obtained from

$$\theta = \sin\theta \sqrt[3]{\sec\theta}$$

is within one second of the truth.

7. Prove that the hyperbola $xy = c^2$ and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ will touch if $2c^2 = ab$, whatever be the angle between the axes, and that they will cut at right angles if the angle between the axes of co-ordinates is

$$\cos^{-1}\left\{c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\}.$$

8. If the normal at α to $\frac{l}{r} = 1 + e\cos\theta$ meet the curve again in the point β, shew that

$$\tan\frac{\beta}{2} = -\cot\frac{\alpha}{2}\,\frac{1 + 2e\cos^2\frac{\alpha}{2} + e^2}{1 - 2e\sin^2\frac{\alpha}{2} + e^2}.$$

9. If the normals at P, Q, R to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet on the fixed normal at the point whose eccentric angle is α, shew that the sides of the triangle PQR touch the conic

$$\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha\right)^2 + 2\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha\right) + 1 = 0.$$

10. Four equal rods AB, BC, CD, DA are freely jointed so as to form a square and are suspended at the corner A. The rods are kept apart by a weightless string joining the middle points of AB, BC. Shew that the tension of the string and the reaction at C are respectively $4W$ and $\frac{1}{2}W\sqrt{5}$.

11. Particles slide from a common vertex down a number of straight tubes each of length l in the same vertical plane. Shew that the locus of the foci of the subsequent parabolic paths consists of portions of two curves whose polar equations may be written

$$r = l\cos\tfrac{1}{2}\theta \quad \text{and} \quad r = l\sin\tfrac{1}{2}\theta,$$

the common vertex being the pole.

12. Supposing the earth to be a homogeneous sphere rotating with uniform angular velocity about a diameter, shew that at a point whose angular distance from a pole is α, the plumb-line will make an angle θ with the normal to the surface which is given by the equation

$$\tan\theta = \frac{\sin\alpha\cos\alpha}{c - \sin^2\alpha},$$

where $\left(1 - \frac{1}{c}\right)$ is the ratio of the force of gravity at the equator to that at a pole.

XXVIII.

1. Shew that the angle between the pedal lines of any two points on a circle is half the angle subtended by those points at the centre: also that the pedal lines of the extremities of any diameter intersect on the nine-point circle.

2. A parabola touches a given straight line at a given point, and its axis passes through a second given point. Shew that the envelope of the tangent at the vertex is a parabola, and determine its focus and directrix.

3. Prove that

$$(1\,.\,2)^2 x + (2\,.\,3)^2 x^2 + (3\,.\,4)^2 x^3 + \ldots = \frac{4x}{(1-x)^3} + \frac{24x^2}{(1-x)^5}.$$

4. Prove that the determinant

$$\begin{vmatrix} a, & b, & c, & b, & a \\ b, & c, & b, & a, & a \\ c, & b, & a, & a, & b \\ b, & a, & a, & b, & c \\ a, & a, & b, & c, & b \end{vmatrix}$$

is equal to

$$(2a + 2b + c)\,(a^2 + b^2 - c^2 + bc + ca - 3ab)^2.$$

5. A, B, C, D are the vertices of a quadrilateral circumscribing a circle of radius R, and Δ, Δ' are the respective areas of $ABCD$, and the quadrilateral whose vertices are the points of contact of $ABCD$ with the circle. Shew that

$$\frac{\Delta}{\Delta'} = \frac{OA\,.\,OB\,.\,OC\,.\,OD}{2R^4},$$

where O is the centre of the circle.

6. If $\sin^{-1}(x + iy) = \tan^{-1}(\xi + i\eta)$, prove that

$$(\xi^2 + \eta^2)^2 \{(x-1)^2 + y^2\} \{(x+1)^2 + y^2\} = (x^2 + y^2)^2.$$

7. From a point P on a parabola two normals are drawn to the curve. Prove that the bisectors of the angles between these form, with the diameter through P and the normal at P, a harmonic pencil.

8. A triangle is circumscribed to the circle $x^2+y^2=r^2$, and two of its angular points lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Prove that the third angular point lies on the ellipse

$$\frac{x^2}{(a^2r^2+b^2r^2-a^2b^2)^2}+\frac{y^2}{(a^2r^2+b^2r^2+a^2b^2)^2}=\frac{r^2}{(a^2r^2-b^2r^2+a^2b^2)^2}.$$

9. Prove that the locus of the centres of all conics which have double contact with a given conic, the chords of contact being in a fixed direction, is the diameter of the given conic which is conjugate to the given direction.

10. A rhombus, formed of four equal freely-jointed rods, is hung over a smooth sphere of radius r, so as to rest symmetrically in a vertical plane. Shew that if a be the length of each rod, the angle it makes with the vertical is a root of the equation

$$\cot^3\theta+\cot\theta=\frac{a}{r}.$$

11. Two equal elastic balls are projected toward each other at the same instant in the same vertical plane, v being the velocity and α the elevation in each case. Shew that after impact they will return to the points of projection if

$$ga(1+e)=ev^2\sin 2\alpha,$$

e being the coefficient of elasticity and $2a$ the distance between the points of projection.

12. A particle hangs attached to one end of an inelastic string of length a, the other end being attached to a fixed point O. The particle is projected so as to move in a vertical circle round O with a velocity due to a height h above O. Prove (1) that if $a>\frac{2}{3}h$, the circular motion will cease when the particle has risen to a height $\frac{2}{3}h$ above O; (2) that the distance from O of the axis of the subsequent free parabolic path is

$$a\left(1-\frac{4h^2}{9a^2}\right)^{\frac{3}{2}}.$$

XXIX.

1. If a circle cuts the sides of a triangle ABC in the points X, X'; Y, Y'; Z, Z', shew that the triangles formed by the lines $Y'Z$, $Z'X$, $X'Y$ and YZ', ZX', XY' are in perspective with ABC, and that the triangles have a common centre of perspective.

2. PSP' is a focal chord of an ellipse, and the tangents at P and P' intersect in T. Prove geometrically that

$$\frac{1}{SP \cdot SP'} - \frac{1}{ST^2} = \frac{1}{BC^2}.$$

3. A bracelet is composed of similar beads of two different sizes, three groups of small ones, each $2n$ in number, being separated by three single larger ones. Shew that the bracelet may be restrung in $3n(n+1)$ distinct ways.

4. Prove that

$$(2n)^{n+1} - n(2n-1)^{n+1} + \frac{n(n-1)}{2!}(2n-2)^{n+1} - \ldots = \frac{3n}{2} \cdot (n+1)!.$$

5. If $\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)$

$$+ \sin(x'-\theta)\sin(y'-\theta)\sin(z'-\theta) = 0,$$

and $$x + y + z + x' + y' + z' = \frac{\pi}{2},$$

then will $$\tan\theta = \frac{\cos x \cos y \cos z + \sin x' \sin y' \sin z'}{\cos x' \cos y' \cos z' + \sin x \sin y \sin z}.$$

6. Shew that

(i) $$\sum_1^\infty \tan^{-1}\left(\frac{2}{n^2}\right) = \frac{3\pi}{4}.$$

(ii) $$\frac{1}{1^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \ldots = \tfrac{1}{3}(\pi^2 - 9).$$

7. A triangle is inscribed in a parabola with its orthocentre at the focus. Prove that its circumscribing circle touches the tangent at the vertex.

8. Shew that the locus of the point such that the tangents from it to the ellipse of semi-major axis a make an angle 2α with each other is given by the equation

$$a_1^2 \cos^2 \alpha + a_2^2 \sin^2 \alpha = a^2$$

where a_1 and a_2 are the primary semi-axes of the confocals through the point.

9. Shew that the director circle of the conic

$$\alpha^2 + 4\beta\gamma \cos A = 0$$

is $\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma (\sec A - 2 \cos A) - \alpha \sec A (\beta \cos B + \gamma \cos C) = 0.$

10. A uniform regular tetrahedron is hung up by three vertical strings attached at three corners A, B, C so that BC is horizontal and below the level of A, and the plane ABC is inclined at an angle 30° to the horizon. Prove that the tensions are as

$$2\sqrt{6} - 1 : 2\sqrt{6} - 1 : 2\sqrt{6} + 2.$$

11. A shot is to be fired so as to enter horizontally, with a given velocity, a wooden partition perpendicular to the plane of its path, and, on emerging, to hit an object at the same level as the gun on the other side. The object is at a distance a and the partition at a distance d from the point of projection, and the thickness of the partition, supposed small, is b. It is found, however, that the shot falls short of the object by a small distance c. Shew that, if the object is to be hit, the thickness of the partition must be diminished by

$$\frac{2(a-d)bc}{a(2d-a)} \text{ approximately,}$$

the resistance of the wood being supposed uniform.

12. An elliptic tube with its major axis vertical rotates with uniform angular velocity about that axis. Shew that a particle inside the tube cannot rest in relative equilibrium except at the lowest point, unless

$$\omega > \left(\frac{g}{l}\right)^{\frac{1}{2}},$$

where $2l$ is the latus-rectum.

XXX.

1. Through one point of intersection of two circles a line is drawn. The points in which it meets the circles are joined to their other point of intersection. Prove that the orthocentre of the triangle so formed lies on a fixed circle.

2. A conic touches the three sides of a triangle and has one focus at the orthocentre. Find the other focus.

3. If $a_1\lambda^2 + b_1\lambda + c_1 = a_2\lambda^2 + b_2\lambda + c_2 = a_3\lambda^2 + b_3\lambda + c_3$, shew that

$$\begin{vmatrix} 1, & b_1, & c_1 \\ 1, & b_2, & c_2 \\ 1, & b_3, & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1, & b_1, & 1 \\ a_2, & b_2, & 1 \\ a_3, & b_3, & 1 \end{vmatrix} = \begin{vmatrix} a_1, & 1, & c_1 \\ a_2, & 1, & c_2 \\ a_3, & 1, & c_3 \end{vmatrix}^2.$$

4. Prove that the fraction most nearly equal to $\sqrt{5}$, but less than $\sqrt{5}$, whose denominator does not exceed 200 is $\frac{360}{161}$.

5. The internal common tangents (other than the sides) to the inscribed and each of the escribed circles of a triangle ABC are drawn. Shew that the vertices of the triangle formed by them lie in the perpendiculars from the centre of the inscribed circle on the sides of ABC: that the area of the triangle is $r^2 \tan A \tan B \tan C$, and that the radius of its circumcircle is

$$\tfrac{1}{4} r \sec A \sec B \sec C.$$

6. Prove that the general solution of the equation

$$\cos z = a \quad (a > 1)$$

is

$$z = 2n\pi + i \cosh^{-1} a.$$

7. If two ellipses have perpendicular latera-recta $2l$, $2l'$ and a common focus S, and touch at a distance c from S, prove that their eccentricities are

$$\sqrt{\left(1 - \frac{l}{l'}\right)\left(1 - \frac{l}{c}\right)} \text{ and } \sqrt{\left(1 - \frac{l'}{l}\right)\left(1 - \frac{l'}{c}\right)}.$$

8. Shew that the equation of the circle of curvature at any point (x', y') of a rectangular hyperbola having the axes of coordinates as asymptotes is

$$x^2 + y^2 - \frac{x}{x'}(3x'^2 + y'^2) - \frac{y}{y'}(3y'^2 + x'^2) + 3(x'^2 + y'^2) = 0,$$

and hence shew that the centre of curvature divides the normal chord externally in the ratio 1 : 3.

9. Prove that the four common tangents to the conics

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0, \quad l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0$$

are the lines

$$\sqrt{ll'(mn' - m'n)}\,\alpha \pm \sqrt{mm'(nl' - n'l)}\,\beta \pm \sqrt{nn'(lm' - l'm)}\,\gamma = 0$$

and that their points of contact lie on the conic

$$ll'(mn' + m'n)\alpha^2 + mm'(nl' + n'l)\beta^2 + nn'(lm' + l'm)\gamma^2 = 0.$$

10. Five rods in the same plane are smoothly jointed together in the form of two triangles ABC, DBC on the same base BC, and on the same side of it, AD being parallel to BC. The middle points of AC and BD are joined by a string at tension T. Shew that the stress in BC is

$$\frac{AD + BC}{2BC}\,T.$$

11. A particle of mass m is attached by two inelastic strings to particles of masses m' and m''. The particles are placed at rest on a smooth horizontal table, so that the two strings lie in perpendicular straight lines. A blow is given to the particle m in the direction of the bisector of the angle between the strings. Shew that the initial velocities of m' and m'' are in the ratio

$$m + m'' : m + m'.$$

12. Four similar rods form a square $ABCD$, fixed on a smooth horizontal table. The coefficient of elasticity between a particle and each of the rods is e. The particle is projected from A so that after striking BC, CD, DA it arrives at B. Shew that the direction of projection makes with AB the angle

$$\tan^{-1}\frac{e + e^2}{1 + e + e^2}.$$

XXXI.

1. Find a point on the circumcircle of a triangle such that its pedal line may be parallel to a given straight line.

2. A, B, C, D, E are five points on a circle. Conics are drawn having E for focus and touching the sides of BCD, CDA, DAB, ABC respectively. Prove that they are parabolas and that their directrices are tangents to another parabola having E for focus.

3. If a be positive and not equal to unity, then

$$\frac{1 + a^2 + a^4 + \ldots + a^{2n}}{a + a^3 + a^5 + \ldots + a^{2n-1}} > \frac{n+1}{n}.$$

4. From three bags each containing tickets numbered from 1 to p, three tickets are drawn, one from each. Shew that the chance that the sum of the numbers on them is $2p$ is

$$\frac{(p-1)(p+4)}{2p^3}.$$

5. If $\alpha + \beta + \gamma = \pi$, and

$$\cos(x + \alpha)\cos(x + \beta)\cos(x + \gamma) + \cos^3 x = 0,$$

then
$$\tan x = \cot\alpha + \cot\beta + \cot\gamma,$$
$$\tan^2 x = 2 + \cot^2\alpha + \cot^2\beta + \cot^2\gamma.$$

6. Shew that the equation

$$\sin x = \tanh x$$

has an infinite number of real roots, and that the large positive roots occur in pairs in the neighbourhood of $x = (2n + \frac{1}{2})\pi$, where n is a large positive integer.

7. Prove that the locus of the point

$$x = \frac{at^2 + bt + c}{At^2 + Bt + C}, \quad y = \frac{a't^2 + b't + c'}{At^2 + Bt + C},$$

where t is a variable parameter, will be a straight line if

$$A(bc' - b'c) + B(ca' - c'a) + C(ab' - a'b) = 0,$$

but that it may be necessary to include imaginary values of t if the point is to trace out the whole line.

For example, find the Cartesian equation of the line

$$x = a\frac{t^2 + 2t - 5}{2t^2 - 6t + 5}, \quad y = a\frac{3t^2 - 8t + 6}{2t^2 - 6t + 5},$$

and shew that real values of t correspond to a length $\sqrt{26}\,.\,a$ of the line.

8. From any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, two lines are drawn touching the rectangular hyperbola $4xy=c^2$, and meeting the ellipse again in Q and R. Prove that the envelope of QR where the position of P is variable is the conic

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)(c^4+a^2b^2)-4c^2xy=a^2b^2.$$

9. Prove that the general equation of a conic inscribed in the quadrilateral $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ may be written

$$(\mu-\nu)^2(\beta\gamma+\alpha\delta)+(\nu-\lambda)^2(\gamma\alpha+\beta\delta)+(\lambda-\mu)^2(\alpha\beta+\gamma\delta)=0.$$

10. A circular disc of radius a and weight W is placed within a smooth sphere of radius b, and a particle of weight w is placed on the disc. Shew that the greatest distance from the centre of the disc at which the particle can rest is

$$\mu\sqrt{b^2-a^2}\left(\frac{W}{w}+1\right).$$

11. The centres of two inelastic balls in contact are B and C. A third ball, centre A, strikes the two simultaneously and in such a way that the direction of motion of A is unchanged by the impact, and there is no impact between B and C. Shew that if β, γ be the angles which the direction of motion of A makes with AB, AC respectively, and if M_2, M_3 be the masses of B and C, then

$$M_2\sin 2\beta=M_3\sin 2\gamma,$$

and that if v be the final, u the initial velocity of A and M_1 its mass, then

$$(M_1+M_2\cos^2\beta+M_3\cos^2\gamma)\,v=M_1u.$$

12. Two balls A and B are moving in the same straight line with equal velocities and B impinges directly on a wall. Shew that there will be at least two impacts between A and B if the ratio of their masses is greater than $2e:1+e^2$, where e is the coefficient of elasticity for the two balls and also for either ball and the wall. Shew also that if the masses are equal, there will be at least three impacts between the balls if $e<2-\sqrt{3}$.

XXXII.

1. Construct a triangle similar to a given triangle, and having a given circle for its nine-point circle.

2. If two chords are drawn from any point on a conic equally inclined to the normal at that point, the tangents at their further extremities will intersect on the normal.

3. Prove that the equation

$$e^x - 1 = 2x$$

has only one positive root, and that its value is about 1·25.

4. If $ax + by + cz = x + y + z$, and the quantities involved are all positive, shew that

$$a^{ax} b^{by} c^{cz} > 1.$$

5. On the sides BC, CA of a triangle segments of circles BOC, COA are described containing angles $\pi - C$, $\pi - A$ respectively: prove that the square of the cosecant of either of the angles OBC, OCA, OAB is equal to

$$\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

6. Sum to n terms the series

(i) $\tanh \alpha + \dfrac{1}{2} \tanh \dfrac{\alpha}{2} + \dfrac{1}{2^2} \tanh \dfrac{\alpha}{2^2} + \ldots,$

(ii) $\sin \theta + 2 \sin 2\theta + 3 \sin 3\theta + \ldots,$

and shew that

$$\cos\theta \frac{\cos\theta}{1} - \sin 2\theta \frac{\cos^2\theta}{2} - \cos 3\theta \frac{\cos^3\theta}{3} + \sin 4\theta \frac{\cos^4\theta}{4} + \cos 5\theta \frac{\cos^5\theta}{5} - \ldots = \cot^{-1}(1 + \tan\theta + \tan^2\theta).$$

7. Shew that the focus of the parabola

$$(x\sqrt{a} + y\sqrt{b})^2 (a+b) + 2(gx + fy)(a+b) + g^2 + f^2 = 0$$

is at the point $\quad -g(a+b)^{-1}, \quad -f(a+b)^{-1}.$

8. Prove that the radius of the circumcircle of the triangle formed by tangents from (h, k) to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and their chord is

$$\frac{1}{2} \frac{\sqrt{b^4h^2 + a^4k^2}}{b^2h^2 + a^2k^2} \rho\rho',$$

where ρ, ρ' are the distances from (h, k) to the foci.

9. Two conics inscribed in the triangle ABC touch the sides in DEF, $D'E'F'$. Shew that if EF, $E'F'$ meet in P; FD, $F'D'$ in Q, and DE, $D'E'$ in R, then AP, BQ, CR are concurrent, and if $(\alpha'\beta'\gamma')$ be the point of concurrence, then the fourth common tangent to the two conics is

$$\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0.$$

10. A circular ring of radius a and weight W is suspended horizontally by three vertical threads, each of length l, attached to points on its circumference. Prove that, in order to hold the ring horizontal, and twisted through an angle θ, a horizontal couple must be applied equal to

$$W \frac{a^2 \sin\theta}{\left(l^2 - 4a^2 \sin^2 \frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

11. Two uniform rods, each of weight W and length a, are hinged together at one end: another uniform rod of weight W' and length $2b$ has small smooth rings at its ends, and these slide freely one along each of the first pair of rods, which are suspended by strings each of length l fastened to the ends of the rods and to a fixed point. The whole system rests symmetrically with the hinge lowest. Prove that the inclination θ of either of the rods W to the vertical satisfies the equation

$$(2W + W')\, a \sin\alpha \cos\theta + (W + W')\, a \cos\alpha \sin\theta = W'b \cos\alpha \operatorname{cosec}^2\theta,$$

where $$\sin\alpha = \frac{a}{l} \sin\theta.$$

12. A smooth tube in the form of a parabola is placed with its axis vertical and vertex downwards and a heavy particle is dropped from the highest point A. If the length of the tube be varied by cutting off portions from the end B, shew that the envelope of the different parabolic paths described after emerging from B is another parabola.

XXXIII.

1. Pairs of harmonic conjugates DD', EE', FF' are taken on the sides BC, CA, AB of a triangle with respect to pairs (BC) etc. Prove that the corresponding sides of the triangles DEF, $D'E'F'$ intersect on the sides of ABC, viz. EF and $E'F'$ on BC, etc.

2. The vertical angle of a cone is 60°, and an elliptic section is taken such that its axis major is at right angles to one of the generating lines which it meets. A circular section is taken through the point where the elliptic section meets the axis of the cone. Prove that the areas of these sections are to each other in the ratio $9\sqrt{3} : 8\sqrt{2}$.

3. Prove that

$$\frac{{}^{2n-1}S_n}{(2n)!} - \frac{{}^{2n-2}S_n}{1!(2n-1)!} + \frac{{}^{2n-3}S_n}{2!(2n-2)!} \dots + \frac{(-1)^{n-1} \cdot {}^{n}S_n}{(n-1)!(n+1)!} = \frac{1}{2^n \cdot n!},$$

where ${}^nS_r$ denotes the sum of the products r together of the first n natural numbers.

4. In a bag there are a number of tickets marked with the natural numbers from 0 to n^2+1. Every number r is marked on each of r tickets, and every square number m^2 confers a prize of m shillings. A person draws one ticket from the bag. Shew that the value of his expectation is

$$\frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)} \text{ shillings.}$$

5. A convex quadrilateral whose sides in order are a, b, c, d is described about a circle: the line joining the points of contact of a and c is of length x, and the line joining the other two points of contact is of length y. Prove that

$$x^2 : y^2 = bd : ac.$$

6. A is a vertex of a regular polygon of n sides inscribed in a circle whose centre is O and radius a. P is the middle point of OA. Prove that the sum of the fourth powers of the distances of the vertices of the polygon from P is $\dfrac{33n}{16}a^4$.

7. Perpendiculars SM, $S'N$ are drawn to the focal distances SP, $S'P$ of any point P on an ellipse to meet the tangent at P in

M and N. Prove that if the eccentricity of the ellipse is not less than $\dfrac{1}{\sqrt{2}}$, the minimum value of the rectangle $PM \,.\, PN$ is $4b^2$, but that otherwise it is $\dfrac{b^2}{e^2(1-e^2)}$.

8. Normal chords are drawn at the extremities of the latera-recta of a series of ellipses confocal with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Shew that they envelop the parabolas $y^2 = \pm\, 4cx$, where $c^2 = a^2 - b^2$.

9. Prove that the area of the triangle formed by two tangents to $S = 0$ and their chord is

$$S^{\frac{3}{2}}\sqrt{-\Delta}/(CS-\Delta),$$

where $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, and $C = ab - h^2$.

10. A ladder leans against a wall with its lower end on the horizontal ground and its upper at a height h above the ground. The angles of friction at the upper and lower ends are η and ϵ respectively, and the inclination of the ladder to the horizontal is θ. A man whose weight is $\dfrac{1}{n}$th that of the ladder slowly ascends. Prove that the greatest height he can reach is either h or

$$h \sec(\epsilon-\eta)\sec\theta\,\{(1+n)\sin\epsilon\sin(\theta+\eta) - \tfrac{1}{2}n\cos\theta\cos(\epsilon-\eta)\},$$

whichever of these two heights be the lesser.

11. A cylinder of weight W' and radius r stands on a rough horizontal table, at a distance c from an edge: over the cylinder is slipped a smooth circular ring of weight w. A rod of length $2l$ and weight W is smoothly jointed to the ring and rests projecting over the edge, while the ring touches the cylinder at two points. Neglecting the friction between the rod and table, prove that $\cos^3\theta = (c/l)(1 + w/W)$, where θ is the inclination of the rod to the horizon. Prove also that in order that the cylinder may not upset $(W + w)\, c \tan^2\theta - wr$ must be less than rW'.

12. Two equal inelastic particles A and B connected by a tight inextensible string rest on a table, and A is constrained to move in a straight horizontal groove inclined at an angle 45° to the string. If a third equal particle C is projected perpendicular to the groove with velocity V so as to strike B directly, shew that A starts off with velocity $\frac{1}{5}V$.

XXXIV.

1. Prove that a circle which touches a given circle and cuts a given straight line at a given angle also touches a second circle, the straight line being the radical axis of the two circles.

2. Triangles are inscribed in an ellipse such that the tangent at each vertex is parallel to the opposite side. Shew that they are all of the same area.

3. Given that for all integral values of n

$$u_{n+1} = u_0 u_n + u_1 u_{n-1} + u_2 u_{n-2} + \dots + u_n u_0,$$

and $u_0 = 1$, $u_1 = 1$, shew that

$$u_n = \frac{1 \,.\, 3 \,.\, 5 \dots (2n-1)}{2 \,.\, 4 \,.\, 6 \dots 2n} \, \frac{2^{2n}}{n+1}.$$

4. If $a_1, a_2, a_3 \dots$ be all positive and $a_1 + a_2 + a_3 \dots$ divergent, then

$$\frac{a_1}{a_1+1} + \frac{a_2}{(a_1+1)(a_2+1)} + \frac{a_3}{(a_1+1)(a_2+1)(a_3+1)} + \dots$$

is convergent and equal to unity.

5. Shew that the area of the triangle whose angular points are the in-centre, circum-centre and ortho-centre of a triangle is

$$-2R^2 \sin \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(C-A) \sin \tfrac{1}{2}(A-B).$$

6. Prove that

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \tfrac{1}{2}\sqrt{7}.$$

7. Prove that the equation to the common tangents to the circles

$$x^2 + y^2 = 2ax \text{ and } x^2 + y^2 = 2by$$

can be expressed in the form

$$2ab\,(x^2 + y^2 - 2by) = (ax - by + ab)^2.$$

8. Through a point P on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ straight lines PQR, $PQ'R'$ are drawn parallel to the asymptotes of the confocal through P, meeting the major axis in Q, Q' and the minor axis in R, R'. Prove that QR', $Q'R$ intersect at a point

on the normal at P, and that the locus of this point is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.$$

9. A circle touches the conic $\frac{l}{r} = 1 + e\cos\theta$ and passes through the pole. Shew that the envelope of the chord joining the other two points of intersection is the conic

$$\frac{l^2}{r^2}\left(\frac{1-e^2}{e^2}\right)^2 + \frac{2l}{r}\frac{1-e^2}{e}\cos\theta + e^2\cos^2\theta = 1.$$

10. The extremities of a rod of length $2b$ and weight w, slide on two equal smooth rods of length $2a$ and weight W, which rest symmetrically in equilibrium over two small pegs in the same horizontal line, the last two rods being smoothly jointed together and the joint above the pegs. Prove that the angle 2θ between the jointed rods is given by

$$\sin^3\theta = \frac{(2W + w)c - bw}{2Wa},$$

where $2c$ is the distance between the pegs and $c > b$.

11. In that system of pulleys in which each hangs by a separate string, there are three moveable pulleys each of weight w and a fixed pulley: round the latter passes the string from which the weight P, the motive power, hangs freely. When the weight W is ascending the first and second strings (reckoning from the bottom) are joined by a new string, and just before the latter becomes tight the old strings are cut so as to release the central moveable pulley. Prove that the velocity of W is increased in the ratio

$$(32P + 9w + W) : (16P + 5w + W).$$

12. A particle P of mass M rests in equilibrium on a smooth horizontal table, being attached to three particles of masses m_1, m_2, m_3 by fine strings which pass over smooth pulleys A, B, C at the edge of the table. Prove that if the string which supports m_3 be cut, the particle will begin to move in a direction making with CP an angle

$$\tan^{-1}\frac{(m_1 - m_2)\{(m_1 + m_2)^2 - m_3^2\}\Delta}{4Mm_1m_2m_3^2 + (m_1 + m_2)\Delta^2},$$

where $$\Delta^2 = 2\Sigma m_2^2 m_3^2 - \Sigma m_1^4.$$

XXXV.

1. Find a point D in the base of a triangle ABC, such that the circles inscribed in the triangles ABD, ACD may touch AD at the same point.

2. If the normal at P to a rectangular hyperbola meet the curve in Q, prove that

$$PQ^2 = 3CP^2 + CQ^2,$$

where C is the centre.

3. Prove that the conditions that u can be so chosen that the equations

$$(a+u)\,x^2 + 2\,(b+u)\,x + (c+u) = 0,$$
$$(a'+u)\,x^2 + 2\,(b'+u)\,x + (c'+u) = 0,$$

may both have real roots are either

$$(a+c-2b)\,(a'+c'-2b') \geqslant 0,$$

or

$$(a+c-2b) > 0 > (a'+c'-2b')$$

together with

$$(b^2-ac)/(a+c-2b) - (b'^2-a'c')/(a'+c'-2b') > 0,$$

or similar conditions with the accented and unaccented letters interchanged.

4. Prove that

$$\frac{(2m-1)\,!}{m\,!\,(m-1)\,!}$$

is an even number, except when m is a power of 2.

5. A quadrilateral of area Δ circumscribes a circle of radius r and centre O, and the lines joining the points of contact intersect at right angles in P. If PO is of length d and the angle OPA is α, where A is one of the points of contact, shew that

$$2d^2r^2 \sin 2\alpha = \{\Delta^2\,(r^2-d^2)^2 - 16r^6\,(r^2-d^2)\}^{\frac{1}{2}}.$$

6. Prove that when n is odd

$$\cot^2\frac{\pi}{n} + \cot^2\frac{2\pi}{n} + \ldots + \cot^2\frac{(n-1)\,\pi}{2n} = \tfrac{1}{6}\,(n-1)\,(n-2).$$

7. Two parabolas touch at P and intersect at Q, R. Prove that PQ, PR are harmonically conjugate to the diameters of the two curves at P.

8. Prove that the equation to the radical axis of the two circles of curvature at the extremities of the diameter of an ellipse inclined to the major axis at an angle θ is

$$b^4 \cos^3\theta \,.\, x - a^4 \sin^3\theta \,.\, y = 0.$$

9. Shew that the co-ordinates of the centre of the rectangular hyperbola passing through four fixed points, referred to the common self-conjugate triangle of all the conics through the four points, are given by

$$\frac{\alpha}{a}(\eta^2 - \zeta^2) = \frac{\beta}{b}(\zeta^2 - \xi^2) = \frac{\gamma}{c}(\xi^2 - \eta^2),$$

where (ξ, η, ζ) is any one of the four points.

10. A framework of weight W is made up of twelve equal freely-jointed rods forming the edges of a parallelopiped. It is suspended freely by one corner, and a smooth sphere of weight w is placed inside it so as to be supported by the three bars which meet in the opposite corner. Shew that the frame will hang so that the inclination θ of the bars to the vertical is given by the equation

$$\left(\frac{W}{2w} + 1\right) \sin^3\theta = \tfrac{1}{3} n \cos\theta,$$

where n is the ratio of the radius of the sphere to the length of a bar.

11. If the angle of elevation of a projectile be $< \cos^{-1}\frac{1}{3}$, its distance from the point of projection will always be increasing: if greater, then after a time it will draw nearer that point and subsequently again recede. In the latter case, prove that the duration of the period of approach is

$$\frac{u}{g}\sqrt{1 - 9\cos^2\alpha},$$

where u is the velocity, and α the angle, of projection.

12. A smooth particle is projected vertically from the vertex and along the concave side of the arc of a parabola whose axis is horizontal, the velocity of projection being that due to the latus-rectum. Shew that if the coefficient of elasticity be $\frac{1}{4}$, the direction of motion after the first rebound from the parabola will be horizontal.

XXXVI.

1. ABC is a triangle and the tangents at A, B, C to its circumscribing circle form the triangle $A'B'C'$. Shew that the circumscribing, polar and nine-point circles of the triangle ABC and the circumcircle of the triangle $A'B'C'$ are coaxial.

2. Having given any three points on a conic and the tangents at two of them, find a construction for the tangent at the third point.

3. Shew that compound interest at the rate of c per cent. per annum payable every instant of time is equivalent to compound interest at the rate of $100\,(e^{\frac{c}{100}} - 1)$ per cent. per annum, payable yearly.

If the interest were payable once a month instead of every instant, prove that the corresponding annual rate would be approximately

$$\left(100 - \frac{c^2}{2400}\right) e^{\frac{c}{100}} - 100.$$

4. If n and $n-2$ be both prime integers, prove that

$$4\,.\,(n-3)\,! + n + 2 \equiv 0 \quad \{\text{mod. } n\,(n-2)\}.$$

5. Shew that the result of eliminating θ between

$$a \cos(\theta + \alpha) + b \cos(\theta + \beta) + c \cos(\theta + \gamma) = 0$$

and $$a \operatorname{cosec}(\theta + \alpha) + b \operatorname{cosec}(\theta + \beta) + c \operatorname{cosec}(\theta + \gamma) = 0$$

is $$\frac{a}{a + b\cos(\alpha - \beta) + c\cos(\alpha - \gamma)} + \frac{b}{a\cos(\beta - \alpha) + b + c\cos(\beta - \gamma)} + \frac{c}{a\cos(\gamma - \alpha) + b\cos(\gamma - \beta) + c} = 0.$$

6. Prove that the equation whose roots are $\tan\left(\frac{1}{15} r\pi\right)$ where r is any number (including unity) less than and prime to 15 is

$$x^8 - 92x^6 + 134x^4 - 28x^2 + 1 = 0.$$

7. Prove that the equation to the chords through (h, k) to the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ such that normals at their extremities are concurrent is $\dfrac{h}{a^2(y-k)} - \dfrac{k}{b^2(x-h)} + \dfrac{1}{hy - kx} = 0.$

8. Through the intersections of the two conics $ax^2 + by^2 = 1$ and $hxy = 1$ a series of conics is drawn. Prove that the locus of their foci is

$$(ax^2 - by^2)(bx^2 - ay^2) = (a - b)(y^2 - x^2) - 2\frac{(a - b)^2}{h}xy.$$

9. The directrix of the parabola which touches the sides of the triangle of reference and also the line $l\alpha + m\beta + n\gamma = 0$, is

$$\alpha \cos A\left(\frac{b}{m} - \frac{c}{n}\right) + \beta \cos B\left(\frac{c}{n} - \frac{a}{l}\right) + \gamma \cos C\left(\frac{a}{l} - \frac{b}{m}\right) = 0.$$

10. A homogeneous sphere of radius a and weight W is placed on a fixed sphere of radius b, the surfaces being sufficiently rough to prevent sliding. The common normal at the point of contact makes an angle α with the vertical. Prove that if a weight w be greater than $W \sin \alpha/(1 - \sin \alpha)$ it may be attached to the surface of the first sphere so that this position may be one of equilibrium. Prove also that the equilibrium can be stable if

$$w > W\lambda/(1 - \lambda),$$

where $$\lambda = \{a^2 \cos^2 \alpha + (a + b)^2 \sin^2 \alpha\}^{\frac{1}{2}}/(a + b).$$

11. Two small rings, each of weight W, slide one upon each of two rods in a vertical plane, each inclined at an angle α to the vertical: the rings are connected by a fine elastic string of natural length $2a$ and modulus λ: the coefficient of friction for each rod and ring is $\tan \beta$. Shew that if the string is horizontal, each ring will rest at any point of a segment of the rod whose length is

$$W\lambda^{-1} a \operatorname{cosec} \alpha \{\cot(\alpha - \beta) - \cot(\alpha + \beta)\}.$$

12. A pile is driven a feet vertically into the ground by n blows of a steam hammer fastened to the head of the pile. Prove that if p is the mean pressure of the steam in pounds per square inch, d the diameter of the piston in inches, l the length of the stroke in feet, w the weight of the moving part of the hammer, and W the weight of the pile and the fixed part of the hammer attached to it, then the mean resistance of the ground in pounds weight is

$$W + w + \frac{nw}{W + w}(w + \tfrac{1}{4}\pi d^2 p)\frac{l}{a}.$$

XXXVII.

1. A, B, C are three circles and P is a circle coaxial with B and C cutting A orthogonally. Q and R are circles similarly obtained from C and A, A and B respectively. Shew that P, Q, R are coaxial.

2. P, Q, R are three points on an ellipse such that the corresponding points on the auxiliary circle form an equilateral triangle. Prove that the normals to the ellipse at P, Q, R are concurrent.

3. Shew that the coefficient of $x^n y^n$ in the expansion in ascending powers of x and y of

$$(1-x)^{-1}(1-y)^{-1}(1-x-y) \text{ is } -1.$$

4. Shew that

(i) $$1+\frac{n}{2}+\frac{n(n+2)}{2\,.\,4}+\frac{n(n+2)(n+4)}{2\,.\,4\,.\,6}+\dots \text{ to } (n+1) \text{ terms}$$
$$=\frac{3}{2}\,\frac{(n+2)(n+4)\dots(3n-2)}{2\,.\,4\,.\,6\dots(2n-2)}.$$

(ii) $$\frac{1}{x}-\frac{n}{x(x+1)}+\frac{n(n-1)}{x(x+1)(x+2)}-\dots$$
$$+\frac{(-1)^n\,.\,n\,!}{x(x+1)\dots(x+n)}=\frac{1}{x+n}.$$

5. Prove that if PL, PM, PN be the perpendiculars from any point P on the sides of the triangle ABC, then

$$PL^2\sin 2A+PM^2\sin 2B+PN^2\sin 2C$$
$$=\Delta\left(\frac{OP^2}{R^2}+4\cos A\cos B\cos C\right),$$

where O is the orthocentre.

6. From any point O on the circumference of a circle of radius a lines are drawn making angles

$$\frac{\pi}{2n},\ \frac{2\pi}{2n}\dots\dots\ \frac{(n-1)\pi}{2n}$$

with the diameter through O. Prove that the product of the lengths intercepted on them by the circumference is $a^{n-1}\,.\,\sqrt{n}$.

7. An ellipse of semi-axes a, b moves so as always to touch the axes of co-ordinates which are rectangular. Prove that the locus of either focus is the curve

$$(x^2+y^2)(x^2y^2+b^4)=4a^2x^2y^2.$$

8. Shew that the directrix of the conic which is described having the origin as focus and osculating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point whose eccentric angle is α is

$$ax\cos^3\alpha - by\sin^3\alpha + \frac{a^2b^2}{a^2-b^2}=0.$$

9. A lever of negligible weight can turn about a fixed point at one end and rests without friction against a fulcrum at $\frac{1}{n}$th of its length from that end, so as to be inclined at an angle α to the horizontal. Shew that when a load P is suspended from the free end, the lever will press against the joint with a force

$$P\{(n-1)^2\cos^2\alpha+\sin^2\alpha\}^{\frac{1}{2}}.$$

10. A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are α and α', and coefficients of friction $\tan\lambda$ and $\tan\lambda'$. Shew that if θ be the limiting inclination to the horizon, W the weight, and R and R' the pressures, then

$$2\tan\theta=\cot(\alpha'+\lambda')-\cot(\alpha-\lambda),$$

$$\frac{R}{\cos\lambda\sin(\alpha'+\lambda')}=\frac{R'}{\cos\lambda'\sin(\alpha-\lambda)}=\frac{W}{\sin(\alpha-\lambda+\alpha'+\lambda')}.$$

11. Two smooth equal spheres of elasticity e are moving with equal velocities in directions including an angle 2α and impinge simultaneously on each other and on two spheres in all respects like the former, lying in contact at rest. At the moment of collision the four centres form a square, and the motion is symmetrical. Prove that the paths of the first pair will diverge at an angle

$$2\tan^{-1}\left(\frac{2e\tan\alpha}{1-e}\right).$$

12. A bead is on a circular wire which rotates uniformly with angular velocity ω about its centre in its own plane which is vertical. If the bead slip shew that it does so at a point whose angular distance from the top is

$$\lambda-\sin^{-1}\left(\frac{\omega^2 a}{g}.\sin\lambda\right),$$

where λ is the angle of friction.

XXXVIII.

1. $ABCD$ is the common diameter of two circles, cutting them in A, B and C, D respectively, and $A'B'C'D'$ the common diameter of the two inverse circles. Prove that the cross-ratios $(ABCD)$ and $(A'B'C'D')$ are equal.

2. If two ellipses have a common focus S and equal major axes, and if one ellipse revolves in its plane about S, the chord of intersection will envelop a conic confocal with the fixed ellipse.

3. If
$$A = 1 + n_3x^3 + n_6x^6 + \ldots,$$
$$B = n_1x + n_4x^4 + n_7x^7 + \ldots,$$
$$C = n_2x^2 + n_5x^5 + n_8x^8 + \ldots,$$
where n_r denotes the number of combinations of n things r together, prove that
$$A^3 + B^3 + C^3 - 3ABC = (1 + x^3)^n.$$

4. Prove that the nth convergent of
$$\frac{1}{2-}\,\frac{2\,.\,3}{7-}\,\frac{4\,.\,5}{11-}\cdots\,-\frac{(2n-2)(2n-1)}{4n-1}\cdots$$
is
$$\frac{1\,.\,3\,.\,5\ldots(2n+1)}{2\,.\,4\,.\,6\ldots 2n} - 1.$$

5. Shew that there are five values of θ not differing by multiples of π which satisfy the equation
$$a \tan 3\theta + b \tan 2\theta + c \tan \theta = d,$$
and if these be α, β, γ, δ, ϵ, shew that
$$\alpha + \beta + \gamma + \delta + \epsilon = n\pi + \tan^{-1}\frac{d}{a+b+c},$$
where n is some integer.

6. Prove that
$$\cot^2\frac{\pi}{11} + \cot^2\frac{2\pi}{11} + \cot^2\frac{3\pi}{11} + \cot^2\frac{4\pi}{11} + \cot^2\frac{5\pi}{11} = 15.$$

7. A finite parabolic arc makes angles θ and ϕ with its chord: prove that the latus-rectum is to the length of the chord in the ratio
$$4\sin^2\theta\sin^2\phi\sin(\theta+\phi) : [4\sin^2\theta\sin^2\phi + \sin^2(\phi-\theta)]^{\frac{3}{2}}.$$

8. T is the pole of a chord PQ of an ellipse whose centre is C, and CT meets the curve in R. Prove that, if the eccentric angles of P and Q are $\alpha+\beta$ and $\alpha-\beta$, the eccentric angle of R is α.

Hence shew that, if p is the length of the perpendicular from the centre on the chord PQ, and if $CR = d$, then

$$\frac{\cos^2\beta}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{d^2}{a^2b^2},$$

a and b being the semi-axes of the ellipse.

9. The general equation of a conic confocal with

$$ax^2 + 2hxy + by^2 = 1$$

is

$$(x^2+y^2)(ab-h^2) + \lambda(ax^2+2hxy+by^2) = \frac{(a+\lambda)(b+\lambda)-h^2}{\lambda}.$$

10. If $(2n+1)$ equal particles each of weight W be connected by means of $(2n+2)$ weightless equal strings each of length l, and hang with the two free extremities attached to two fixed points at a horizontal distance $2a$, shew that the horizontal tension T is given by the equation

$$\frac{a}{l} = \sum_{m=0}^{m=n} \left\{1 + (m+\tfrac{1}{2})^2 \frac{W^2}{T^2}\right\}^{-\frac{1}{2}}.$$

11. To a sphere of mass m are attached by two light strings two other spheres of mass m. The system is placed on a smooth table with the strings tight and at right angles, and a fourth sphere of mass M impinges directly on the first with velocity V. If the system be inelastic and the strings remain tight, shew that the velocity of the impinging sphere after impact is

$$\frac{MV}{M+2m}.$$

12. A ball of mass M lies at the bottom of a circular tube in a vertical plane. A second ball of mass m falling down the tube impinges on it, the coefficient of restitution being m/M. Shew that the heights to which the balls rise in the tube after the second impact are in the ratio

$$M^2 : (M-m)^2.$$

XXXIX.

1. A line is drawn cutting two given circles in the points P, P' and Q, Q' respectively so that the range P, Q, P', Q' is harmonic. Shew that the middle points of PP' and QQ' lie on a fixed circle for all positions of the chord.

2. A parabola touches two lines AP, AQ in P and Q. Any other tangent to the parabola meets these lines in B and C, and BQ, CP meet in R. Prove that the polar of R passes through the centroid of the triangle ABC.

3. Shew that the rational form of the equation

$$x^{\frac{1}{5}} + y^{\frac{1}{5}} + z^{\frac{1}{5}} = 0$$

is

$$\alpha^5 + 625\gamma(5\beta - \alpha^2) = 0,$$

where α, β, γ denote Σx, Σyz and xyz respectively.

4. Prove that

$$\sum_{r=1}^{r=n} (-1)^{r-1}\, {}^nC_r \,.\, 2^{n-r} \,.\, r^3 = n(n^2 - 6n + 6).$$

5. The distances of any point O in the plane of the triangle ABC from A, B, C are $\frac{x}{a}\sqrt{2R}$, $\frac{y}{b}\sqrt{2R}$, $\frac{z}{c}\sqrt{2R}$. Shew that

$$16a^2b^2c^2\sigma(\sigma - x)(\sigma - y)(\sigma - z) = \{R \,.\, \Sigma a^2(y^2 + z^2 - x^2) - a^2b^2c^2\}^2,$$

where $2\sigma = x + y + z$, and R is the radius of the circumcircle.

6. A regular polygon $A_1A_2 \ldots A_n$ is inscribed in a circle of radius a. Any point P is taken in the plane of the circle distant c from the centre. Prove that

$$\sum_{r=1}^{r=n} PA_r^2 = n(a^2 + c^2),$$

$$\Sigma\Sigma PA_r^2 \,.\, PA_s^2 = \frac{n(n-1)}{2}(a^2 + c^2)^2 - na^2c^2 \qquad (r \neq s).$$

7. Shew that the point midway between the centres of curvature at the extremities of conjugate diameters of an ellipse traces the curve

$$(ax + by)^{\frac{2}{3}} + (ax - by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

8. Prove that the product of the lengths of the tangents from the point (x, y) to the conic

$$u \equiv ax^2 + 2hxy + by^2 = 1$$

is

$$\left(1-\frac{1}{u}\right)C^{-1}\{C^2(x^2+y^2)^2+(a-b)^2+4h^2+2C(a-b)(x^2-y^2)+8Chxy\}^{\frac{1}{2}},$$

where $C=ab-h^2$.

9. If the equation $\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$ represent a parabola, the equation of its axis is

$$\frac{a^2\alpha}{l}\left(\frac{b^4}{m^2}-\frac{c^4}{n^2}\right)+\frac{b^2\beta}{m}\left(\frac{c^4}{n^2}-\frac{a^4}{l^2}\right)+\frac{c^2\gamma}{n}\left(\frac{a^4}{l^2}-\frac{b^4}{m^2}\right)=0.$$

10. A hexagon $ABCDEF$ formed of six equal heavy rods, each of weight W, is supported at A and is prevented from collapsing by two light rods BE and CF which are supposed to pass freely across each other. Shew that the thrust in each of these rods is $3W$.

11. Two masses m_1, m_2 connected by a fine string are suspended over a smooth weightless pulley. This pulley is connected by a string passing over a fixed pulley with a mass M hanging freely. Shew that m_1 will remain at rest if

$$M=\frac{4m_1m_2}{3m_2-m_1}.$$

Under these circumstances m_2 after descending for T seconds strikes an inelastic plane and remains at rest. Shew that the string over the weightless pulley will be tight again after t seconds where

$$t=\frac{2}{3}\frac{m_2-m_1}{m_2}T.$$

12. A ball is projected from the lower edge of one of the walls of a room with a square floor of side a feet at an inclination θ to the horizon with a velocity which would, if vertical, just carry it to the ceiling. Prove that after hitting the other three walls in succession it will return to the point of projection if

$$\sin 2\theta=\frac{a(1+e)}{2e^2h}\left\{1+e^2-2\frac{c}{a}(1-e)+\frac{c^2}{a^2}(1-e)^2\right\}^{\frac{1}{2}},$$

where h is the height of the room, and c the distance of the point of projection from the wall on which the ball first impinges.

XL.

1. A, B, C, D are four points on a circle. BC and AD intersect at E, CA and BD at F, AB and CD at G. Prove that the remaining intersection of the circles ABE, ADG is the foot of the perpendicular from F on EG.

2. Shew that two conics can be drawn through four given points A, B, C, D to touch a given line, and that if D be inside the triangle ABC, the conics can only be real when the straight line divides two of the sides of ABC internally and one externally.

3. Solve the equations

(i) $(1+x+x^2+x^3)^2+3(1+x^2+x^4+x^6)=0$,

(ii) $\Sigma x^2=21,\quad \Sigma yz=-10,\quad \Sigma y^2z^2=84.$

4. Prove that the nth convergent of the continued fraction

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\ldots$$

is

$$\left|\begin{array}{cccccc} 2, & 1, 0, 0\ldots & 0, & 0, 0 \\ -1, & 3, 1, 0\ldots & 0, & 0, 0 \\ 0, & -1, 4, 1\ldots & 0, & 0, 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0, & 0, 0\ldots\ldots & -1, & n-1, 1 \\ 0, & 0, 0\ldots\ldots & 0, & -1, n \end{array}\right| \div \left|\begin{array}{cccc} 1, & 1, 0, 0\ldots & 0, & 0, 0 \\ -1, & 2, 1, 0\ldots & 0, & 0, 0 \\ 0, & -1, 3, 1\ldots & 0, & 0, 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0, & 0, 0\ldots\ldots & -1, & n-1, 1 \\ 0, & 0, 0\ldots\ldots & 0, & -1, n \end{array}\right|.$$

5. Shew that

$$(2\cos a)(2\cos 2a)(2\cos 2^2a)\ldots(2\cos 2^{n-1}a)$$
$$=2\{\cos a+\cos 3a+\cos 5a+\ldots+\cos(2^n-1)a\}.$$

6. If m be positive, the sums of the three series

$$1-\frac{m(m-1)(m-2)}{3\,!}+\ldots,$$

$$m-\frac{m(m-1)(m-2)(m-3)}{4\,!}+\ldots,$$

$$\frac{m(m-1)}{2\,!}-\frac{m(m-1)(m-2)(m-3)(m-4)}{5\,!}+\ldots$$

are respectively

$$\frac{2}{3} \cdot 3^{\frac{m}{2}} \cos \frac{m\pi}{6},\ \frac{2}{3} \cdot 3^{\frac{m}{2}} \cos (m-2) \frac{\pi}{6} \text{ and } \frac{2}{3} \cdot 3^{\frac{m}{2}} \cos (m-4) \frac{\pi}{6}.$$

7. Shew that the vertices of a triangle self-conjugate with regard to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be written

$$(a \cos \beta \cos \gamma \sec \alpha,\ b \sin \beta \sin \gamma \operatorname{cosec} \alpha),$$
$$(a \cos \gamma \cos \alpha \sec \beta,\ b \sin \gamma \sin \alpha \operatorname{cosec} \beta),$$
$$(a \cos \alpha \cos \beta \sec \gamma,\ b \sin \alpha \sin \beta \operatorname{cosec} \gamma).$$

8. Prove that the lines

$$\sqrt{c-2b} \,.\, x \pm \sqrt{b} \,.\, y = \frac{ac}{b} \sqrt{c-2b}$$

are normal to both the parabolas

$$y^2 = 4ax \text{ and } y^2 = 4(a+b)(x+c).$$

9. If the chords of curvature of a rectangular hyperbola, whose centre is C, at Q_1, Q_2, Q_3, Q_4 all pass through P, prove that

$$CQ_1^2 + CQ_2^2 + CQ_3^2 + CQ_4^2 = CP^2.$$

10. Two equal uniform rods, rigidly jointed at an angle α, are suspended from one extremity. Shew that the action of the joint consists partly of a force, and partly of a couple of magnitude

$$3Wa \sin \alpha / (10 - 6 \cos \alpha)^{\frac{1}{2}},$$

where W is the weight, and $2a$ the length of either rod.

11. Two inelastic spheres of equal radius and masses $\lambda_1 m$, $\lambda_2 m$ are lying in contact upon an inelastic horizontal plane. A third sphere of the same radius and mass m falls freely with its centre in the vertical plane containing the centres of the other two spheres, so as to impinge upon them simultaneously. Prove that the velocity produced in one of the spheres which was at rest is

$$V\sqrt{3}\,(1 + 2\lambda_2)/(1 + 4\lambda_1 + 4\lambda_2 + 12\lambda_1\lambda_2),$$

where V is the velocity of the falling sphere just before impact.

12. A heavy particle is placed on the surface of a smooth fixed sphere of radius r at an angular distance α from the highest point. Shew that the latus-rectum of the parabola described after leaving the sphere is

$$\tfrac{16}{27} r \cos^3 \alpha.$$

XLI.

1. ABC, DBC are two equilateral triangles on the same base BC and P is any point on the circle whose centre is D and radius DB or DC. Prove that PA, PB, PC are the sides of a right-angled triangle.

2. Prove that any diameter of a central conic and the reflexion in either axis of the conjugate diameter form with the equiconjugate diameters a harmonic pencil.

3. If there be n pairs of things, those in any one pair being alike, prove that the number of permutations of the $2n$ things taken r together is $r\,!$ times the coefficient of x^r in the expansion of

$$(1 + x + \tfrac{1}{2}x^2)^n.$$

4. Shew that if $2^n + 1$ be a prime number, and a an odd number not divisible by $2^n + 1$, then $a^{2^n} - 1$ is divisible by $2^n(2^n + 1)$. Shew also that if c be prime to 10, then $c^{20} - 1$ is divisible by 200.

5. If R, r, ρ be the radii of the circumscribed, inscribed and polar circles of a triangle, prove that the area of the triangle is

$$r(r + 2R + \rho)^{\frac{1}{2}}(r + 2R - \rho)^{\frac{1}{2}}.$$

6. If $\theta = \dfrac{2\pi}{33}$, prove that

$$\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta + \cos 16\theta = \frac{1 + \sqrt{33}}{4},$$

$$\cos 5\theta + \cos 10\theta + \cos 13\theta + \cos 7\theta + \cos 14\theta = \frac{1 - \sqrt{33}}{4}.$$

7. Shew that the centres of curvature at the points where $y = mx$ meets conics confocal with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ lie on the curve

$$y\{x^2 - 3mxy - a^2 + b^2\}^2 + mx\{m(y^2 + a^2 - b^2) - 3xy\}^2 = 0.$$

8. Two conics have a common focus, equal latera-recta, and four real points of intersection. Prove that one of them must be a hyperbola, and that if the other be an ellipse the sum of the

reciprocals of the distances of the common points from the common focus is

$$\frac{4}{l} \cdot \frac{e'^2 - ee' \cos \gamma}{e'^2 + e^2 - 2ee' \cos \gamma},$$

where $2l$ is the latus-rectum, e and e' the eccentricities of the ellipse and hyperbola respectively, and γ the angle between the axes.

9. A triangle ABC is circumscribed to a conic whose foci are S and S', and p, q, r are the perpendiculars (with proper signs) from A, B, C on any variable tangent. Shew that

$$\frac{AS \,.\, AS' \,.\, BC}{p} + \frac{BS \,.\, BS' \,.\, CA}{q} + \frac{CS \,.\, CS' \,.\, AB}{r} = 0.$$

10. Three smooth straight rods making equal angles with each other form a rigid framework, which is fixed at O so that the rods make equal angles α with the downward vertical. A weight W is suspended by three strings of equal length a which are fixed one to each of three small smooth rings of weight w sliding on the rods. Prove that in the position of equilibrium the depth of W below the rings is

$$aW/\{W^2 + (W + 3w)^2 \cot^2 \alpha\}^{\frac{1}{2}}.$$

11. An endless elastic string of natural length $2\pi r$ and modulus λ is placed lightly on a smooth surface in the form of a paraboloid of revolution with its axis vertical and vertex upwards, so that it forms a horizontal circle, and is then let go. Prove that when it first comes to rest on the surface its radius will be

$$\frac{4\lambda\pi a + Wr}{4\lambda\pi a - Wr} \cdot r,$$

where W is the weight of the string, and $4a$ the latus-rectum.

12. At the top of a straight tube of length l inclined to the horizon is a horizontal platform of length a. A particle is projected up the tube with velocity $\sqrt{\frac{5}{2}gl}$. Shew that it cannot possibly clear the platform if $4a > 3\sqrt{3}l$, and that if $4a = 3\sqrt{3}l$, it will just clear it if the inclination of the tube to the horizon is 30°.

XLII.

1. Through a given point two circles are drawn cutting at a given angle and touching a given circle. Prove that the locus of their other point of intersection is a circle coaxial with the given circle and the given point.

2. The circle of curvature of an ellipse at P touches the tangent at the point diametrically opposite to P. Prove that the diameter conjugate to CP is a mean proportional between the axes of the ellipse.

3. Prove that

$$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}\equiv\frac{1}{n}+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}\right)$$
$$+\frac{1}{3}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right)+\frac{1}{4}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}\right)+\dots.$$

4. Four men play a game as follows: each puts £1 into the pool and the first man who cuts an ace from a pack of cards (the cards being replaced and shuffled after every cut) takes the pool. Shew that the expectation of the man who cuts first is about £1. 2*s.* 6*d.*

5. If $A+B+C=A'+B'+C'=\pi$, then

$$\sin^2 B\sin^2 C'+\sin^2 C\sin^2 B'-2\sin B\sin B'\sin C\sin C'\cos(A+A')$$

is equal to two similar expressions and also to

$$1-\Sigma\cos^2 A\cos^2 A'+2\cos A\cos B\cos C\cos A'\cos B'\cos C'$$
$$+2\sin A\sin B\sin C\sin A'\sin B'\sin C'.$$

6. Prove that the determinant

$$\begin{vmatrix} \cos\theta, & 1, & 0, & 0, & \dots & 0 \\ 1, & 2\cos\theta, & 1, & 0, & \dots & 0 \\ 0, & 1, & 2\cos\theta, & 1, & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1, & 2\cos\theta, & 1 \\ \dots & \dots & \dots & 0, & 1, & 2\cos\theta \end{vmatrix}$$

of order n is equal to $\cos n\theta$.

7. Prove that the equation $\cot x = ax$, where a is real and positive, has no imaginary roots.

8. The rectangular co-ordinates of a point on a conic are given by

$$\frac{x}{at^2+bt+c}=\frac{y}{a't^2+b't+c'}=\frac{1}{a''t^2+b''t+c''},$$

where t is a variable parameter. Prove that the eccentricity is given by

$$\frac{e^4}{1-e^2}+4=\frac{(AC'+A'C-B^2-B'^2)^2}{(AC-B^2)(A'C'-B'^2)-\frac{1}{4}(AC'+A'C-2BB')^2},$$

where A, B, C... are the minors of a, b, c... in the determinant

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

9. At a point (x', y') of a rectangular hyperbola $x^2-y^2=a^2$ is drawn the parabola of four-point contact. Prove that the equation to its directrix is

$$2xx'+2yy'=x'^2+y'^2.$$

10. A heavy uniform rod is supported by two strings of lengths l and l', one at each end, the other ends of the strings being tied to two weightless rings which are free to slide on two smooth fixed rods in a vertical plane inclined at angles α, β to the horizon. If there is equilibrium when the rings are in a horizontal line, prove that the length of the rod is

$$(l\cos\alpha-l'\cos\beta)\operatorname{cosec}\left\{\tan^{-1}\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}\right\}.$$

11. A man stands at the upper end of a long rough plank of length a and mass M which lies along a smooth groove on an inclined plane and has its upper end supported by a cord. The cord is cut, and at the same instant the man starts off and runs with very short steps down the plank at such a rate that the plank does not move. Prove that the velocity of the man at the lower end is

$$\left(2ga\cos\alpha\,.\,\frac{m+M}{m}\right)^{\frac{1}{2}},$$

where m is the mass of the man, and α the inclination of the groove to the vertical.

12. A railway train is travelling round a curve of large radius a with uniform speed V. A pendulum of length l is hung from the roof of one of the carriages. Shew that approximately it will oscillate in the same time as an ordinary pendulum of length

$$l\left(1-\frac{1}{2}\frac{V^4}{a^2g^2}\right).$$

XLIII.

1. Two circles A and B meet in two points O and O'. A third circle meets A in P, Q and B in R, S. Prove that

$$OP \,.\, OQ : O'P \,.\, O'Q = OR \,.\, OS : O'R \,.\, O'S.$$

2. If O is the centre of curvature at the point P of an ellipse, and R be taken so that $OP = PR$, shew that P and R are conjugate points with respect to the director circle.

3. If $x + y + z = 0$, and none of the quantities x, y, z vanish and no two are equal, prove that

$$\left(\frac{x^3}{y-z} + \frac{y^3}{z-x} + \frac{z^3}{x-y}\right)\left(\frac{y-z}{x^3} + \frac{z-x}{y^3} + \frac{x-y}{z^3}\right) + 12xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^3 = 0.$$

4. If an experiment succeeds 3 times out of 4, shew that the chance that in n consecutive trials there are never three consecutive successes is

$$(\tfrac{3}{4})^{n+1}(2 + \alpha^{n+1} + \beta^{n+1}),$$

where α and β are the roots of the equation $3x^2 + 2x + 1 = 0$.

5. If

$$\begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma & \sin\delta \\ \cos\alpha & \cos\beta & \cos\gamma & \cos\delta \\ \sin\beta & \sin\alpha & \sin\delta & \sin\gamma \\ \cos\beta & \cos\alpha & \cos\delta & \cos\gamma \end{vmatrix} = 0,$$

then either one of the differences $\alpha - \beta$, $\gamma - \delta$ is a multiple of π, or else $\alpha + \beta - \gamma - \delta$ is an even multiple of π.

6. Prove the formula

$$\sqrt{2} = \frac{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots}.$$

7. Prove that the tangent to the conic $\dfrac{l}{r} = 1 + e\cos\theta$ parallel to the tangent at α is

$$\frac{l}{r} = \frac{e^2 - 1}{e^2 + 2e\cos\alpha + 1}\{\cos(\theta - \alpha) + e\cos\theta\}.$$

8. Shew that the equation to the four normals from (x_0, y_0) to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\{a^2(x-x_0)^2+b^2(y-y_0)^2\}(xy_0-x_0y)^2=(a^2-b^2)^2(x-x_0)^2(y-y_0)^2.$$

9. Shew that the angle between the tangents from any point to the general conic $S=0$ is

$$\tan^{-1}\frac{2\sqrt{-\Delta S}}{C(x^2+y^2)-2Gx-2Fy+A+B},$$

and if the conic be a hyperbola the product of the perpendiculars from any point on the asymptotes is

$$\left(S-\frac{\Delta}{C}\right)\Big/\{(a-b)^2+4h^2\}^{\frac{1}{2}},$$

where S, Δ, A, B, etc. have their usual meanings.

10. The two horizontal sides of a square consist of two uniform rods of length $2a$, of which the upper one is fixed: the two vertical sides consist of two strings. A rough plane is raised into contact with the lower rod, which it touches only at the ends and is turned, remaining horizontal, so that the centres of the rods remain in a vertical line. Shew that, if the coefficient of friction is $\frac{1}{\sqrt{3}}$, when the equilibrium of the lower rod is limiting the tension of either string must be at least $W\sqrt{3}/4\sqrt{2}$, where W is the weight of the lower rod.

11. A light string ABC is fixed at A and particles of masses m, m' respectively are fastened at B and C, and the whole system is held in a vertical plane so that AB, BC make acute angles α, $\alpha+\beta$ with the vertical. Prove that when B and C are released, the initial tension of AB is

$$\frac{m(m+m')g\cos\alpha}{m+m'\sin^2\beta}.$$

12. A particle is dropped from a railway bridge so as just to alight on the front carriage of a train passing underneath. The height of the bridge above the top of the train is h, the velocity of the train is that due to a height k, the coefficient of friction between the particle and the roof of the carriage is $\frac{2}{3}$, and the coefficient of restitution is $\frac{1}{2}$. Shew that the particle will not hit the train a second time if the length of the train is less than $2h^{\frac{1}{2}}(k^{\frac{1}{2}}-h^{\frac{1}{2}})$.

XLIV.

1. The line joining two given points A and B cuts a given plane in C: through C any line in the plane is drawn, and on this line a point P is taken so that $AP + PB$ is a minimum. Shew that the locus of P for all positions of the line is a circle.

2. Through a point P three normals are drawn to a parabola, and the circle through the points in which they meet the parabola again has its centre on the axis of the parabola. Shew that the locus of P is an equal parabola.

3. If m and n are positive integers and $m > n > 1$, then

$$\left(1+\frac{1}{m}\right)^m > \left(1+\frac{1}{n}\right)^n > 2^n\left(\frac{n!}{n^n}\right)^{\frac{3}{2}}.$$

4. The fifth power of a number is of the form $91m + 5$. Shew that the number is of the form $91m + 31$.

5. Shew that the expression

$$\sin^{2n}\alpha\sin(\beta+\gamma)\sin(\beta-\gamma)+\sin^{2n}\beta\sin(\gamma+\alpha)\sin(\gamma-\alpha)$$
$$+\sin^{2n}\gamma\sin(\alpha+\beta)\sin(\alpha-\beta)$$

is divisible by

$$\sin^4\alpha\sin(\beta+\gamma)\sin(\beta-\gamma)+\sin^4\beta\sin(\gamma+\alpha)\sin(\gamma-\alpha)$$
$$+\sin^4\gamma\sin(\alpha+\beta)\sin(\alpha-\beta),$$

and find the quotients in the cases $n = 3$ and $n = 4$.

6. Prove that the area of the triangle, the sides of which are

$$a(1-\kappa^2\sin^2\theta)^{\frac{1}{2}},\quad b\{1-\kappa^2\sin^2(\theta+C)\}^{\frac{1}{2}},\quad c\{1-\kappa^2\sin^2(\theta-B)\}^{\frac{1}{2}},$$

is $(1-\kappa^2)^{\frac{1}{2}}\Delta$, where Δ is the area of the triangle whose parts are a, b, c, A, B, C.

7. If a conic have double contact with $ax^2 + by^2 = 1$ along a normal chord, and also pass through the origin, the locus of its centre is

$$\{a^4x^4 + 2ab(a^2-ab+b^2)x^2y^2 + b^4y^4\}^2 = ab(a-b)^2x^2y^2(a^3x^2+b^3y^2).$$

8. If ρ, ρ' are the radii of curvature and n, n' the normal chords at the extremities of two conjugate semi-diameters of an ellipse of axes $2a$, $2b$, prove that

$$\frac{\rho}{n}+\frac{\rho'}{n'}=\frac{a^4+b^4}{2a^2b^2}.$$

9. Prove that the normals to the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

at the three points of reference will meet in a point if

$$\frac{l}{a}(m^2 - n^2) + \frac{m}{b}(n^2 - l^2) + \frac{n}{c}(l^2 - m^2) = 0.$$

10. A see-saw consists of a plank of weight w laid across a fixed rough log whose shape is that of a horizontal circular cylinder. The inclination to the horizontal at which it balances is increased to α when loads W, W' are placed at the lower and higher ends respectively, and the inclination is reduced to β when the loads are interchanged. Shew that the inclination when unloaded is

$$\frac{w'(W + W' + w)(W'\alpha - W\beta)}{w(W + W' - w')(W - W')},$$

w' being the load which, placed at the higher end, would balance the plank horizontally.

11. A perfectly elastic ball is projected vertically with velocity v_1 from a point in a rigid horizontal plane, and when its velocity is v_2 an equal ball is projected vertically from the same point, also with velocity v_1. Shew

(i) that the time that elapses between successive impacts of the two balls is v_1/g,

(ii) that the heights at which they take place are alternately

$$\frac{(3v_1 - v_2)(v_1 + v_2)}{8g} \text{ and } \frac{(3v_1 + v_2)(v_1 - v_2)}{8g},$$

(iii) that the velocities of the balls at the impacts are equal and opposite, and alternately $\frac{1}{2}(v_1 - v_2)$ and $\frac{1}{2}(v_1 + v_2)$.

12. A particle of mass m is tied by a string of length l to the vertex of a right circular cone of mass M and vertical angle 2α placed on a perfectly rough horizontal table. Prove that the greatest angular velocity with which the particle can be projected without tilting the cone is

$$\left(\frac{g}{l}\right)^{\frac{1}{2}}\left[1 + \left(1 + \frac{M}{m}\right)^2 \tan^2\alpha\right]^{\frac{1}{4}}.$$

XLV.

1. Two circles touch one another internally at A, and a variable chord PQ of the outer touches the inner: shew that the locus of the in-centre of the triangle APQ is another circle touching the given circles at A.

2. Prove that the locus of the middle point of the common chord of a parabola and its circle of curvature at any point is another parabola whose latus-rectum is one-fifth of that of the given parabola.

3. Shew that the number of integral solutions of the equation

$$x + y + 2z = n,$$

including zero, is

$$\tfrac{1}{8}\{2n^2 + 8n + 7 + (-1)^n\}.$$

4. If x is less than unity, prove that the infinite continued fraction

$$\frac{1}{1-}\,\frac{x}{1-}\,\frac{x^2(1-x)}{1-}\,\frac{x^3(1-x^2)}{1-}\,\frac{x^4(1-x^3)}{1-}\cdots$$

is equal to the infinite product

$$(1+x)(1+x^2)(1+x^3)\ldots\ldots$$

5. If θ, ϕ, ψ are real angles which neither are equal nor differ by a multiple of π, and if

$$\begin{aligned}\cos(\phi+\psi)\cos(\phi+\alpha)\cos(\psi+\alpha) + a\cos 2\theta \\ = \cos(\psi+\theta)\cos(\psi+\alpha)\cos(\theta+\alpha) + a\cos 2\phi \\ = \cos(\theta+\phi)\cos(\theta+\alpha)\cos(\phi+\alpha) + a\cos 2\psi,\end{aligned}$$

shew that either

$$\theta + \phi + \psi + \alpha = n\pi, \quad a = 0,$$

or

$$\theta + \phi + \psi + \alpha = (n + \tfrac{1}{2})\pi, \quad a = \tfrac{1}{2},$$

where n is an integer.

6. Sum the series

$$\sum_{r=0}^{r=n-1} \frac{1}{a + b\cos\left(\theta + \dfrac{2r\pi}{n}\right)},$$

where n is any positive integer.

7. A triangle is inscribed in a parabola so that the focus is the centre of the inscribed circle. Prove that the diameter of this circle is $\sqrt{2}-1$ times the latus-rectum of the parabola.

8. Find the equation of the parabola of closest contact at (x', y') on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and shew that its axis is

$$\frac{x}{x'}-\frac{y}{y'}=\frac{a^2-b^2}{x'^2+y'^2},$$

and that its latus-rectum is

$$\frac{2a^2b^2}{(x'^2+y'^2)^{\frac{3}{2}}}.$$

9. The general conic meets the sides of the triangle of reference in A_1A_2, B_1B_2 and C_1C_2. The tangents at A_1, A_2 meet in P, those at B_1, B_2 in Q, and those at C_1, C_2 in R. Shew that AP, BQ, CR meet in the point (F^{-1}, G^{-1}, H^{-1}), where F, G, H have their usual meanings.

10. A weight W is hung from B by two equal wires AB, BC, having their other ends attached to fixed points A, C in the same horizontal straight line. Shew that B is by the extension lowered a distance approximately equal to

$$\tfrac{1}{2} W \,.\, AB/E \cos^2 \alpha,$$

where E is Hooke's modulus for the wires, and 2α the angle ABC.

11. A particle falls vertically with velocity u on one face of a smooth wedge resting with another face in contact with a smooth horizontal plane. Shew that if the mass of the wedge be half that of the particle and α the angle of the wedge, the velocity of the wedge after the particle has struck it n times is

$$\frac{u \sin 2\alpha}{2-\cos 2\alpha} \cdot \frac{1+e}{1-e}(1-e^n).$$

12. Two particles are describing the same parabola under gravity. Shew that the intersection of their directions of motion moves as a heavy particle in an equal coaxial parabola, the distance between the vertices of the two parabolas being $\frac{1}{8}g\tau^2$, where τ is the interval between the instants at which the two particles pass the vertex.

XLVI.

1. If a curve be reciprocated with respect to a circle whose centre is O, and PV, $P'V'$ be the chords of curvature through O at corresponding points P, P' of the curve and its reciprocal, prove that

$$PV\,.\,P'V' = 4OP\,.\,OP'.$$

2. Shew that the area of the greatest parabolic section which can be cut from a given right circular cone is $\frac{\sqrt{3}}{2} r\sqrt{r^2+h^2}$, where r is the radius of the base, and h the height.

3. If
$$\frac{x_1}{a_1+\alpha_1} + \frac{x_2}{a_2+\alpha_1} + \dots + \frac{x_n}{a_n+\alpha_1} = 1,$$
$$\frac{x_1}{a_1+\alpha_2} + \frac{x_2}{a_2+\alpha_2} + \dots + \frac{x_n}{a_n+\alpha_2} = 1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{x_1}{a_1+\alpha_n} + \frac{x_2}{a_2+\alpha_n} + \dots + \frac{x_n}{a_n+\alpha_n} = 1,$$

shew that
$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} = 1 + (-1)^{n-1}\frac{\alpha_1\alpha_2\dots\alpha_n}{a_1a_2\dots a_n}.$$

4. Shew that if p be a prime, then
$$(3p)\,! - 6\,(p\,!)^3$$
is a multiple of p^6.

5. If
$$\sin(x+iy)\sin(\xi+i\eta) = 1,$$
shew that
$$\tan\xi = \pm\frac{\sin x}{\sinh y}, \quad \text{and} \quad \tanh\eta = \mp\frac{\cos x}{\cosh y}.$$

6. A regular polygon of n sides is inscribed in a circle of radius a. A straight line is drawn in the plane of the polygon at a distance $a\sec\alpha$ from the centre of the circle: prove that the sum of the reciprocals of the distances of the angular points of the polygon from the straight line is

$$\frac{n\cot\alpha}{a}\left\{\frac{(1+\sin\alpha)^n - (1-\sin\alpha)^n}{(1+\sin\alpha)^n - 2\cos^n\alpha\cos n\theta + (1-\sin\alpha)^n}\right\},$$

where θ is the angle the radius through one of the angular points of the polygon makes with a perpendicular to the given straight line.

7. If P be any point on a circle passing through the centres of the three circles escribed to the triangle ABC, prove the relation

$$\Sigma (AP^2/bc)(1 + \cos A - \cos B - \cos C) = 1 + \cos A + \cos B + \cos C.$$

8. Prove that the latus-rectum of the parabola which touches the sides AB, AC of a triangle ABC at the points B, C is

$$\frac{4p \sin^2 A \sin B \sin C}{(\sin^2 A + 4 \cos A \sin B \sin C)^{\frac{3}{2}}},$$

where p is the perpendicular from A on BC.

9. If the position of a point be determined by its distances x, y, z from the vertices of a triangle ABC, then the equation

$$lx^2 + my^2 + nz^2 = 2R(-mna^2 - nlb^2 - lmc^2)^{\frac{1}{2}},$$

where $l + m + n = 0$, represents a tangent to the circumcircle.

10. Three equilateral triangles of jointed rods AOB, COD, EOF in one plane are loosely jointed together at the point O, and the rhombuses $OBPC$, $ODQE$, $OFRA$ are completed by six other rods loosely jointed at their extremities. Shew that if stresses are induced by a stretched string connecting P and Q and a rod connecting Q and R, the stress in the rod will be equal to the tension of the string.

11. From a point between two smooth parallel vertical walls c feet apart and in a vertical plane perpendicular to both walls, a particle of elasticity e is projected towards one of the walls, which is distant a feet from the point. After $2n$ impacts with the walls, the particle returns to the point of projection: shew that if α be the angle of projection and V the initial velocity, then

$$V^2 \sin 2\alpha = \frac{g\{c - a(1-e)\}(1 - e^{2n})}{e^{2n} - e^{2n+1}}.$$

12. Two particles of masses p, q respectively are connected by a straight string which passes through a smooth fixed ring. The whole is on a smooth horizontal table, and the particles are projected perpendicularly to the string. Prove that the initial curvatures of their paths are

$$\frac{q}{p+q} \cdot \frac{bu^2 + av^2}{abu^2} \text{ and } \frac{p}{p+q} \cdot \frac{bu^2 + av^2}{abv^2},$$

where u, v are the initial velocities, and a, b the portions of the string.

XLVII.

1. Given two pairs of conjugate rays of an involution pencil, shew how to construct the ray conjugate to a fifth given ray.

2. A, B, C, D are four points on a parabola, and through C and D lines are drawn parallel to the axis, meeting DA, BC in P and Q respectively. Prove that PQ is parallel to AB.

3. If $$(1+x+x^2+\ldots+x^{2m})^n = a_0 + a_1x + a_2x^2 + \ldots,$$
prove that
$$a_2 + 3a_3 + 6a_4 + 10a_5 + \ldots = \frac{mn(2m+1)^n}{3\,!}(3mn+m-2).$$

4. If all the letters denote positive integers, prove that
$$(a^{r+s+1}+b^{r+s+1}+c^{r+s+1})(a^{r+s+3}+\ldots+\ldots)\ldots\ldots$$
$$(a^{r+s+2n-1}+\ldots+\ldots) > (a^{n+s}+\ldots+\ldots)^{n+r}(a^s+\ldots+\ldots)^{-r}.$$

5. A square is circumscribed to a convex quadrilateral. Prove that its area is
$$\frac{x^2y^2-4\Delta^2}{x^2+y^2-4\Delta},$$
where Δ is the area of the quadrilateral and x, y its diagonals.

6. If n be an odd integer, shew that
$$\frac{n\tan n\theta - \tan\theta}{\tan\theta} = \sum_{r=1}^{r=\frac{1}{2}(n-1)} \frac{2\sec^2\frac{(2r-1)\pi}{2n}}{\tan^2\frac{(2r-1)\pi}{2n} - \tan^2\theta}.$$

7. Shew that the value of the continued fraction
$$\frac{1}{1-}\,\frac{1^4}{5-}\,\frac{2^4}{13-}\cdots\frac{n^4}{(2n^2+2n+1)-}\cdots \text{ ad inf.}$$
is $\frac{1}{6}\pi^2$.

8. Prove that the locus of the intersection of normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which are equidistant from the centre consists of the line at infinity, the axes and the two curves
$$r^2(a\cos^2\theta \pm b\sin^2\theta) = (a\mp b)^2(a\pm b)\cos^2 2\theta.$$

9. If θ be the angle between the asymptotes of a conic represented by the general trilinear equation of the second degree, prove that

$$4 \cdot \begin{vmatrix} 0 & \sin A & \sin B & \sin C \\ \sin A & a & h & g \\ \sin B & h & b & f \\ \sin C & g & f & c \end{vmatrix}$$

$$-(a+b+c-2f\cos A-2g\cos B-2h\cos C)^2\tan^2\theta=0.$$

10. If the greatest possible cube be cut out of a solid hemisphere of uniform density, prove that the remainder can rest with its curved surface on a perfectly rough inclined plane of angle α, with its base inclined to the horizon at an angle

$$\sin^{-1}\left\{\frac{8(3\pi-\sqrt{6})}{9\pi-8}\sin\alpha\right\}.$$

11. In a truck of mass M is fixed a vertical tube of small radius, inside which is fixed a particle of mass m. The truck is made to slide on a smooth horizontal table by a horizontal string which, passing over a pulley, is attached to a mass M' which hangs freely. Motion ensues for time t, after which the particle is allowed to fall down the tube. Prove that the particle will describe in space a parabola whose latus-rectum is

$$\frac{2M'^2(M+M'+m)gt^2}{\{(M+M'+m)^2+M'^2\}^{\frac{3}{2}}}.$$

12. A fountain is formed by a jet of water issuing horizontally with velocity u from, and at right angles to, a horizontal tube revolving uniformly about a vertical axis. Prove that every drop of water lies on the surface of a paraboloid of revolution about the vertical axis, and of latus-rectum $\dfrac{2u^2}{g}$.

XLVIII.

1. A point moves so that the squares of the tangents from it to three circles are in Arithmetical progression. Prove that its locus is a straight line forming a harmonic pencil with the radical axes of the circles taken in pairs.

2. A quadrilateral circumscribes a parabola. Shew that the three circles which have the diagonals as diameters cut the directrix in the same two points.

3. Prove the formulae

(i) $^nC_1 - {}^nC_2 \,.\, {}^3C_2 + {}^nC_3 \,.\, {}^4C_2 - {}^nC_4 \,.\, {}^5C_2 \ldots \pm {}^nC_n \,.\, {}^{n+1}C_2 = 0,$

(ii) $$\frac{1}{2p+1} \,.\, {}^{2p}C_p + \frac{1}{2p-1} \,.\, {}^2C_1 \,.\, {}^{2p-2}C_{p-1} + \frac{1}{2p-3} \,.\, {}^4C_2 \, {}^{2p-4}C_{p-2} + \ldots + {}^{2p}C_p = 4^p \,.\, \frac{2 \,.\, 4 \,.\, 6 \ldots 2p}{3 \,.\, 5 \,.\, 7 \ldots (2p+1)}.$$

4. In the continued fraction

$$\frac{1}{1-x+}\,\frac{x}{1-x^3+}\,\frac{x^3}{1-x^5+}\,\frac{x^5}{1-x^7+}\ldots$$

the nth convergent is $\dfrac{\sigma_n}{\sigma_n - 1}$, where

$$\sigma_n = x^{-1^2} - x^{-2^2} + x^{-3^2} - x^{-4^2} + \ldots + (-1)^{n-1} x^{-n^2}.$$

5. In a triangle whose orthocentre is P and circumcentre O, prove that one of the four triangles IOP, I_1OP, I_2OP, I_3OP will be equal to the sum of the other three.

6. Prove that the sum of the infinite series

$$1 + \frac{\cos 3\theta}{3\,!} + \frac{\cos 6\theta}{6\,!} + \ldots$$

is

$$\frac{1}{3}\left[e^{\cos\theta}\cos(\sin\theta) + e^{-\cos\left(\frac{\pi}{3}-\theta\right)} \,.\, \cos\left\{\sin\left(\frac{\pi}{3}-\theta\right)\right\} + e^{-\cos\left(\frac{\pi}{3}+\theta\right)} \,.\, \cos\left\{\sin\left(\frac{\pi}{3}+\theta\right)\right\}\right].$$

7. Shew that the squares of the lengths of the tangents from the point (h, k) to the parabola $y^2 = 4ax$ are the roots of the equation

$$a^2t^2 - (k^2 - 4ah)(2a^2 + k^2 - 2ah)\,t + (k^2 - 4ah)^2 (a^2 + k^2 - 2ah + h^2) = 0.$$

8. Prove that the equation

$$(al^2 + 2hlm + bm^2) S = \{l(ax + hy + g) + m(hx + by + f)\}^2$$

represents a pair of parallel tangents to $S = 0$.

Deduce that the foci are given by

$$aS - (ax + hy + g)^2 = bS - (hx + by + f)^2$$
$$= \{hS - (ax + hy + g)(hx + by + f)\}/\cos\omega,$$

the axes of co-ordinates being inclined at an angle ω.

9. I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC: U is a conic inscribed in the triangle, S one of its foci. Two conics are drawn, one through S, B, C, I, I_1 and the other through S, B, C, I_2, I_3. Shew that their fourth intersection is the second focus of U.

10. A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if

$$\frac{W}{w} > \frac{b - 2a}{a},$$

and that if W/w be equal to this ratio, the equilibrium is really stable.

11. At a horizontal distance a from a gun there is a wall of height $h\ (> a - ga^2/v^2)$. Shew that if the shot be fired off with velocity v in a vertical plane at right angles to that of the wall, there will be a distance

$$\frac{2ha}{g(a^2 + h^2)} \cdot \sqrt{v^4 - a^2g^2 - 2hv^2g}$$

on the other side of the wall commanded by the gun, provided this expression is real.

12. A perfectly elastic particle falls from rest and strikes a fixed smooth plane of inclination α. If the second rebound occurs at an assigned point on the plane, prove that the locus of the point from which the particle falls is a straight line which makes an angle

$$\tan^{-1}(\tan\alpha + \tfrac{1}{4}\operatorname{cosec} 2\alpha)$$

with the horizon.

XLIX.

1. Shew how to bisect a given finite straight line by means of a compass only.

2. The tangent and normal at a point P of a conic meet one axis in T, G and the other in T', G'. Prove that the line joining the middle points of GT, $G'T'$ bisects the radius of curvature at P.

3. If $3u_{r-2} + 10u_{r-1} + 3u_r = 16(r-1)$ for all values of r from 3 to n and also $9u_1 + 3u_2 = 16$, $3u_{n-1} + 5u_n = 8n - 3$, then

$$u_r = r + \tfrac{1}{3}\{(-\tfrac{1}{3})^{r-1} + (-\tfrac{1}{3})^{2n-r-1}\}.$$

4. Prove that

$$e^x\left(x - \frac{x^2}{2!\,2} + \frac{x^3}{3!\,3} - \dots\right) = x + \frac{1+\frac{1}{2}}{2!}x^2 + \frac{1+\frac{1}{2}+\frac{1}{3}}{3!}x^3 + \dots.$$

5. Prove that

(i) $$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

(ii) $$\left(\tan^2\frac{\pi}{7} + \tan^2\frac{3\pi}{7}\right)^{-1} + \left(\tan^2\frac{3\pi}{7} + \tan^2\frac{5\pi}{7}\right)^{-1} + \left(\tan^2\frac{5\pi}{7} + \tan^2\frac{\pi}{7}\right)^{-1} = \frac{17}{26}.$$

6. If $s_n = \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \dots$ ad inf., prove the relation

$$s_{2n+1} = \frac{\pi^2}{4}\left(\frac{1}{2!}s_{2n-1} - \frac{1}{4!}\frac{\pi^2}{2^2}s_{2n-3} + \frac{1}{6!}\frac{\pi^4}{2^4}s_{2n-5} - \dots\right)$$

and hence shew that

$$s_3 = \frac{\pi^3}{32}, \quad s_5 = \frac{5\pi^5}{1536}.$$

7. The product of the latera-recta of the two parabolas which can be drawn through four concyclic points is

$$\tfrac{1}{8}(d_1^2 - d_2^2)\sin\omega,$$

where d_1, d_2 are the diagonals of the quadrilateral and ω the angle between them.

8. If $C \equiv x^2 + y^2 + 2ax + c^2$, $C' \equiv x^2 + y^2 + 2bx + c^2$, then the general equation of all conics having double contact with the circles $C = 0$, $C' = 0$ is

$$(C - C')^2 - 2k(C + C') + k^2 = 0$$

and the envelope of the asymptotes of these conics is the parabola

$$y^2 + 2x(a + b) = 0.$$

9. Prove that the radius of curvature at the point of the general conic where the tangent is parallel to the axis of x is

$$\pm \frac{(-\Delta)^{\frac{1}{2}}}{a^{\frac{3}{2}}}.$$

10. At the corners of a light triangular wire are fastened particles whose masses are proportional to the opposite sides. The particles are placed on an inclined plane with BC horizontal and A tied by a string to a point above it on the same line of greatest slope. Shew that if the plane be tilted gradually about the horizontal, the wire will begin to turn about the orthocentre when the inclination of the plane is

$$\tan^{-1}\{2\mu \sin A/(\cos B - \cos C)\}.$$

11. A gun fires a shot at a target on the same horizontal level and distant a from the gun. The shot just clears a hill which is in the line of fire and afterwards strikes the target. If the hill is of height h and its base is at distance c from the gun, and if the mass of the gun is n times that of the shot, prove that the gun must have been elevated at an angle

$$\tan^{-1}\left\{\frac{n}{n+1} \cdot \frac{ah}{c(a-c)}\right\}.$$

12. A sphere of radius $2a$ and mass M slides inside a fixed smooth hemispherical basin of radius $6a$, starting from the level of the rim of the basin, and impinges on another sphere of radius a and mass m resting at the bottom of the basin. Prove that if $25m = 16M$, and the coefficients of restitution are all unity, the second sphere will just rise to the level of the rim of the basin, and that at every impact the spheres exchange kinetic energies.

L.

1. Shew that the circles described on the diagonals of a quadrilateral as diameters cut orthogonally the four circles with respect to which the triangles formed by the four lines are self-polar.

2. Construct the centre and axes of a conic which passes through five given points.

3. Prove that the sum of the cubes of the coefficients of $(1+x)^n$ is equal to the coefficient of $x^n y^n$ in the expansion of

$$\{1+y(1+x)^2(1+xy)\}^n.$$

4. If $2n$ things be divided into n pairs, shew that the number of ways in which they may be redivided so that no couple of things which were together in the first division shall be together in the second is

$$\phi(n) - n\phi(n-1) + \frac{n(n-1)}{2!}\phi(n-2) \ldots + (-1)^{n-1} n\phi(1) + (-1)^n,$$

where $$\phi(n) = 1 \,.\, 3 \,.\, 5 \ldots (2n-1).$$

5. The sides of a triangle are respectively $52\sqrt{2}$, $56\sqrt{2}$, $60\sqrt{2}$. Prove that the areas of the three triangles formed by joining its angular points to the centre of the nine-points circle are 837, 882, 969.

6. Prove that

$$\frac{1}{1^4 \,.\, 3^4} + \frac{1}{3^4 \,.\, 5^4} + \frac{1}{5^4 \,.\, 7^4} + \ldots = \frac{\pi^4 + 30\pi^2 - 384}{768}.$$

7. If ρ_1, ρ_2, ρ_3 be the radii of curvature of a parabola at the feet of the normals from a point P, prove that $\rho_1\rho_2\rho_3 = 8SP^3$, where S is the focus.

8. An equilateral triangle is inscribed in the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2.$$

If its vertices be (x_1, y_1), (x_2, y_2), (x_3, y_3) and its centroid (x, y) prove that

$$3xy = x_1y_1 + x_2y_2 + x_3y_3.$$

9. Through four points there can in general be described two conics of given eccentricity, but if only one can be described, it is either a rectangular hyperbola, or its eccentricity e satisfies the equation

$$e^4 \tan^2 \alpha + 4e^2 - 4 = 0$$

where α is the angle between the axes of the two parabolas through the four points.

10. A uniform rod of weight W has one end freely hinged and the other supported by an elastic string of natural length a which is tied to a point in the same horizontal with the highest point of the rod at a distance c from it. If in the position of equilibrium the rod makes an acute angle θ with the horizon, and the length of the string is b, prove that the modulus of elasticity is

$$\frac{1}{2} \cdot \frac{aW}{c} \cot \theta \Big/ \left(1 - \frac{a}{b}\right).$$

11. A smooth wedge of angle α and mass M is free to move in a direction perpendicular to its edge on a smooth horizontal table. A particle of mass m is projected from a point in its lower edge up its face at an inclination β to that edge. Prove that the particle will describe a parabola with its axis inclined to the horizon at an angle $\tan^{-1}\left(1 + \frac{m}{M}\right) \tan \alpha$, and that the highest point attained is at a height

$$\frac{M^2 m^2 H V^2 \cos^4 \alpha \sin^2 \beta}{2(M+m) K^4 g}$$

above the vertex, where $H = M + m \sin^2 \alpha$, $K^2 = H^2 + m^2 \sin^2 \alpha \cos^2 \alpha$, and V is the velocity of projection.

12. A particle is projected with velocity V from the cusp of an inverted cycloid down the arc. Shew that the time of reaching the vertex is

$$2(a/g)^{\frac{1}{2}} . \tan^{-1} \{(4ag)^{\frac{1}{2}}/V\}.$$

PART II.

LI.

1. If a tangent be drawn to the circle inscribed in a square, cutting the sides (produced if necessary) in P, Q, R, S, then the rectangles $PQ\,.\,RS$ and $PS\,.\,RQ$ will be equal.

2. Describe a parabola which shall touch three given straight lines, one of them at a given point.

3. Resolve the following expressions into factors:

(i) $(y+z-2x)^5+(z+x-2y)^5+(x+y-2z)^5$.

(ii) $(ax+by)(bx+cy)(cx+dy)(dx+ay)$
$$-(bx+ay)(cx+by)(dx+cy)(ax+dy).$$

4. Prove that the sum of the cubes of all positive fractions in their lowest terms which are less than the integer n and which have 5 for their denominator is

$$\tfrac{1}{5}n^2(5n^2-1).$$

5. If α, β, γ be the roots of the cubic $x^3+ax+b=0$, then the equation whose roots are $\beta^2+\gamma^2-\alpha^2$, $\gamma^2+\alpha^2-\beta^2$, $\alpha^2+\beta^2-\gamma^2$ is

$$x^3+2ax^2+8b^2=0.$$

6. Through the centre of the inscribed circle lines are drawn parallel to the sides a, b, c of a triangle, meeting the sides. If the parts cut off by the sides are a', b', c' shew that

$$\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}=2.$$

7. Prove that

$$\operatorname{cosec} 10^\circ+\operatorname{cosec} 50^\circ-\operatorname{cosec} 70^\circ=6.$$

8. If the tangents be drawn to the parabola $y^2 = 4ax$ from a point P and the corresponding normals meet in a point Q such that PQ cuts the axis at a fixed point O within the curve at a distance k from the vertex, shew that P lies on the circle

$$x^2 + y^2 - x(a+k) + a(2a-k) = 0.$$

9. Shew that the rectangular hyperbola which cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at an angle α, and has the principal axes of the ellipse for asymptotes, is

$$xy = \frac{a^2 b^2 \cos \alpha}{\sqrt{(a^2+b^2)^2 \sin^2 \alpha + 4a^2b^2 \cos^2 \alpha}}.$$

10. Forces P, $2P$, $3P$, $4P$, $5P$, $6P$ act along the sides of a regular hexagon taken in order. Prove that their resultant is parallel to a diagonal of the hexagon, and at a distance from this diagonal equal to $\frac{7}{4}\sqrt{3}a$, where a is the length of a side.

11. An equilateral wedge of mass M is placed on a smooth horizontal table with one of its lower edges in contact with a smooth vertical wall. A smooth ball of mass M' is placed between the wall and the wedge so that motion ensues without rotation. Shew that the ball will descend with acceleration

$$\frac{3M'}{M + 3M'} g.$$

12. Trace the curve $xy^3 = a^4$, and shew that the subtangent at any point of the curve is three times the abscissa.

LII.

1. Shew that if G be the centroid of a triangle ABC the tangents drawn from A, B, C respectively to the circles BGC, CGA, GAB are all equal.

2. Given a parabolic arc, supposed to be entirely on one side of the axis, continue the curve.

3. Prove that the sum of the products of the first n natural numbers taken three together is

$$\tfrac{1}{48}n^2(n^2-1)(n^2-n-2).$$

4. A gardener, having obtained a cutting of a rare plant, takes from it 6 cuttings yearly after it has been growing two years. If he treats each cutting in the same way after it has been growing two years, prove that during the nth year of the growth of the original plant, the number of his plants and cuttings will be

$$\tfrac{1}{5}\{3^n-(-2)^n\}.$$

5. Solve the equations:

$$\text{(i)}\quad 3x^3+2x^2+7x-20=0.$$

$$\text{(ii)}\quad x^4+3x^3-2x^2-7x+3=0.$$

$$\text{(iii)}\quad x^5-10x^2+15x-6=0.$$

6. $ABCDEF$ is a regular hexagon. A circle is drawn touching AB and AF, and a circle of equal radius touches this circle and the sides CD, DE. A third circle touches these two circles and also the side EF. Prove that its radius is $\dfrac{6+5\sqrt{3}}{52}$ times a side of the hexagon.

7. Prove that

$$\tan^2\theta+\tan^2\left(\theta+\frac{\pi}{3}\right)+\tan^2\left(\theta+\frac{2\pi}{3}\right)=9\tan^2 3\theta+6.$$

8. A triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is such that the points on the auxiliary circle corresponding to its angular points form a triangle of given species with angles α, β, γ. Prove that the locus of the centre of gravity of the triangle is the similar ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{9}(1-8\cos\alpha\cos\beta\cos\gamma).$$

9. Shew that the locus obtained by producing the radii vectores drawn from (x_0, y_0) to the general conic in the constant ratio $k:1$ is

$$\begin{aligned}k^2(ax_0^2+2hx_0y_0+by_0^2+2gx_0+2fy_0+c)&\\ +2k\{(x-x_0)(ax_0+hy_0+g)+(y-y_0)(hx_0+by_0+f)\}&\\ +a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2&=0.\end{aligned}$$

10. A piece of board in the form of a square ruled into 16 equal squares is suspended at the middle point of one of its sides, and rests in equilibrium. If one of the bottom corner squares be cut away, prove that the board will turn through an angle $\tan^{-1}\frac{1}{19}$ before assuming its new position of equilibrium.

11. A particle of mass m is attached to one of mass M by a string of length $a+b$, and to another of mass m' by a string of length $c\ (<b)$. The particles m and m' lie close together on a horizontal table at a distance b from the edge while M is held at the edge, the whole system being in a vertical plane perpendicular to the edge. If M be now let go, prove that m' will leave the table with velocity due to a height

$$\frac{M^2a+M(M+m+m')b+[(M+m)^2+mm']c}{(M+m+m')^2}.$$

12. If $y=e^x\tan^{-1}x$, prove that

$$(1+x^2)\frac{d^2y}{dx^2}-2(1-x+x^2)\frac{dy}{dx}+(1-x)^2y=0.$$

LIII.

1. ABC, $A'B'C'$ are two triangles such that the perpendiculars from A on $B'C'$, B on $C'A'$, and C on $A'B'$ are concurrent. Shew that the same is true of the perpendiculars from A' on BC, B' on CA and C' on AB.

2. If a number of ellipses be described on the same major axis, one and the same parabola will pass through the extremities of the major axis and of the latera-recta.

3. Shew that

$$1 + 2(1-x) + 3(1-x)(1-2x) + \dots + n(1-x)(1-2x)\dots(1-\overline{n-1}x)$$
$$\equiv x^{-1}\{1-(1-x)(1-2x)\dots(1-nx)\}.$$

4. Evaluate the determinants

$$\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}.$$

5. If a and b are positive and $a > b$, shew that the roots of the equation

$$x^3 + x^2 - ax - b = 0$$

are all real.

6. Prove that if

$$\cos(4x-y-z)\sin(y-z) + \cos(4y-z-x)\sin(z-x) + \cos(4z-x-y)\sin(x-y) = 0,$$

and no two of the three quantities x, y, z are equal or differ by a multiple of π, then

$$\cos 2x + \cos 2y + \cos 2z = 0.$$

7. In the ambiguous case, given b, c and B, prove that the distance between the orthocentres of the two triangles is

$$2\,(b^2 \cot^2 B - c^2 \cos^2 B)^{\frac{1}{2}}.$$

8. From any point on a parabola a chord is drawn cutting the curve at an angle ϕ and the axis of the parabola at an angle θ. Prove that the length of the chord varies as

$$\sin\phi \operatorname{cosec}(\theta - \phi) \operatorname{cosec}^2\theta.$$

9. Find the foci and eccentricity of the conic

$$5x^2 + 4xy + 8y^2 - 12x - 12y = 0.$$

10. A right circular cone of height h and vertical angle $2a$ is placed with its vertex in contact with a smooth vertical wall, and its slant side resting against a smooth rail parallel to the wall and at distance c from it. Shew that if in the position of equilibrium the axis makes an angle θ with the horizon, then

$$3h = 4c \sec^2(\theta - a) \sec\theta.$$

11. A string fixed at one end passes under a smooth pulley which carries a weight P and then over a fixed pulley, and carries a weight Q hanging vertically. Prove that the acceleration of the centre of mass of the system is always downwards and equal to

$$\frac{(P-2Q)^2}{(P+Q)(P+4Q)} \cdot g.$$

12. Integrate the functions

$$\frac{1}{x^3 - x^2}, \quad \sec^4 x, \quad x^2 \log x \quad \text{and} \quad x\,(a^2 + x^2)^{\frac{3}{2}}.$$

LIV.

1. Four circles are described, each passing through two successive vertices of a cyclic quadrilateral. Prove that the four remaining intersections of successive circles lie on a circle.

2. If TP, TQ be the tangents at the points P, Q to a parabola whose focus is S, prove that if $SP + SQ$ is constant, the locus of T is a parabola, and determine its latus-rectum.

3. Sum to n terms the series

(i) $\frac{1\,!}{r\,!} + \frac{2\,!}{(r+1)\,!} + \frac{3\,!}{(r+2)\,!} + \dots,$

(ii) $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \dots.$

4. Prove that the product of the sum of n positive quantities and the sum of their reciprocals cannot be less than n^2.

5. Prove that the equations

$$ax^3 + 3bx^2 + d = 0, \quad bx^3 + 3dx + e = 0$$

will have a common root if

$$(ae - 4bd)^3 = 27\,(ad^2 + b^2e)^2.$$

6. Prove that, if a and b be the shortest distances from A and B to the straight road on which P and Q lie, and if α, α_1, α_2 be the angles which PQ, PB, QB respectively subtend at A, and β, β_1, β_2 the angles which PQ, PA, QA respectively subtend at B, then

$$a/b = \sin\alpha \sin\beta_1 \sin\beta_2/\sin\beta \sin\alpha_1 \sin\alpha_2.$$

7. If $\tan 3\alpha - \tan\alpha = \tan 3\beta - \tan\beta = \tan 3\gamma - \tan\gamma$, and no two of the angles α, β, γ differ by a multiple of π, then each of these expressions is equal to $2\tan\alpha\tan\beta\tan\gamma$ and $\alpha+\beta+\gamma$ is an odd multiple of $\frac{\pi}{2}$.

8. Shew that the centroids of the triangles of which the three perpendiculars lie along the lines $y = m_1x$, $y = m_2x$, $y = m_3x$ lie on the line

$$y(3 + m_2m_3 + m_3m_1 + m_1m_2) = x(m_1 + m_2 + m_3 + 3m_1m_2m_3).$$

9. Shew that the square of the diameter of the circumcircle of the triangle formed by the lines

$$ax^2 + 2hxy + by^2 = 0, \quad lx + my + 1 = 0$$

is
$$\frac{\{(a-b)^2 + 4h^2\}(l^2 + m^2)}{(am^2 - 2hlm + bl^2)^2}.$$

10. A uniform triangular lamina ABC rests between two smooth pegs D and E in the same horizontal line, its vertex A being downwards. Shew that, if ϕ be the angle which the bisector of the angle at A makes with the vertical in the position of equilibrium, then

$$\frac{3DE}{BC} = \frac{\sin\frac{1}{2}(B-C)}{\sin\phi} \cdot \sin^2\frac{A}{2} + \frac{\cos\frac{1}{2}(B-C)}{\cos\phi} \cdot \cos^2\frac{A}{2}.$$

11. Particles of masses m, m' are attached to the ends of a fine inextensible string. The string passes round a smooth pulley of mass M on a smooth horizontal table, and the ends hang over the edge. The whole system being allowed to move freely from a position of instantaneous rest in which the free parts of the string are perpendicular to the edge of the table, prove that the acceleration of M is

$$\frac{4mm'g}{M(m+m') + 4mm'}.$$

12. Shew that if (α, β) be the intercepts made by the tangent at any point of the curve

$$x^3 + y^3 - 3axy = 0$$

on the axes, the co-ordinates (x, y) of the point of contact are given by

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1, \quad xy = \frac{2a^2\alpha\beta}{\alpha\beta - a^2}.$$

LV.

1. Construct a triangle having given the base, the point where the perpendicular from the vertex meets the base, and the length of a side of the inscribed square.

2. AA' is any diameter of a rectangular hyperbola and PP' is a chord perpendicular to AA'. Shew that the circle $PP'A$ touches the hyperbola at A.

3. Prove that the number of combinations of $3n$ things taken n at a time, of which n are alike and the rest unlike, is the coefficient of x^n in the expansion of $(1+x)^{2n}(1-x)^{-1}$.

4. Sum the infinite series:

(i) $$1+\frac{1}{10^2}+\frac{1.3}{1.2}\cdot\frac{1}{10^4}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6}+\dots,$$

(ii) $$1+2.\frac{1}{10^2}+\frac{3}{1.2}\cdot\frac{1}{10^4}+\frac{4}{1.2.3}\cdot\frac{1}{10^6}+\dots.$$

5. Solve the equation

$$\begin{vmatrix} x & a & b & c \\ c & x & a & b \\ b & c & x & a \\ a & b & c & x \end{vmatrix}=0.$$

6. Prove the identities

(i) $$\Sigma\cos 3\alpha\sin(\beta-\gamma)=4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\cos(\alpha+\beta+\gamma),$$

(ii) $$\sin 9\theta=\sin\theta\,(4\cos^2\theta-1)(64\cos^6\theta-96\cos^4\theta+36\cos^2\theta-1).$$

7. If $x=\sin 2\alpha\sin^2(\beta+\gamma)$, $\quad y=\sin 2\beta\sin^2(\gamma+\alpha)$,

$$z=\sin 2\gamma\sin^2(\alpha+\beta),$$

and $$\tan\alpha+\tan\beta+\tan\gamma=0,$$

prove that $$(x+y+z)^3=27xyz.$$

8. If the sum or difference of the tangents to two circles from a point is equal to the distance between their centres, prove that the locus of the point is a parabola.

9. A circle passes through the centre of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Shew that the product of the perpendiculars from the centre of the conic on a pair of chords of intersection of the conic and the circle is equal to

$$\frac{a^2b^2}{a^2 - b^2}.$$

10. The resultant of three forces acting along the sides a, b, c of a triangle acts along the line joining the centres of the inscribed and circumscribed circles. Shew that the forces are in the ratio

$$(b-c)(b+c-a) : (c-a)(c+a-b) : (a-b)(a+b-c).$$

11. Three elastic particles of masses m_1, m_2, m_3 lie in a straight line on a horizontal table, and m_1 is projected towards m_2. If

$$(m_1 + m_2)(m_2 + m_3)e = m_1 m_3 (1+e)^2,$$

shew that the velocity of m_1 after striking m_2 will be equal to that of m_2 after striking m_3.

12. If

$$\frac{x^2}{c+\lambda} + \frac{y^2}{\lambda} = 1, \quad \frac{x^2}{c-\mu} - \frac{y^2}{\mu} = 1,$$

prove that

$$\frac{\partial\lambda}{\partial x}\cdot\frac{\partial\mu}{\partial y} - \frac{\partial\lambda}{\partial y}\cdot\frac{\partial\mu}{\partial x} = \frac{4cxy}{\lambda+\mu}.$$

LVI.

1. Shew that if O be any point on the circumcircle of the triangle ABC, and OL be drawn parallel to BC to meet the circumcircle in L, then LA will be perpendicular to the pedal line of O with respect to the triangle.

2. The tangents to any conic section at the ends of the latus-rectum meet any ordinate PP' in Q and Q'. Shew that the circle on QQ' as diameter intercepts on the latus-rectum a chord equal to PP'.

3. The first two terms of a harmonic series are a and $a+b$, where b is small compared with a. Shew that the nth term is approximately

$$a+(n-1)b+(n-1)(n-2)\frac{b^2}{a}.$$

4. Shew that the coefficient of x^p in the expansion of

$$\frac{(x+1)(x+2)\dots(x+n)}{(x-1)(x-2)\dots(x-n)} \text{ is } \sum_{r=1}^{r=n}(-1)^{n-r+1}\frac{(n+r)!}{n!\,(r-1)!}\frac{{}^nC_r}{r^{p+1}}.$$

5. The sum of the twentieth powers of the roots of the equation

$$x^4+ax+b=0 \text{ is } 50a^4b^2-4b^5.$$

6. If $A+B+C=m\pi$, where m is an integer, then

$$\frac{\Sigma \sin A \cos A \tan(B-C)}{\Sigma \cos^2 A \tan(B-C)}+\tan A \tan B \tan C=0.$$

7. Prove that

$$2^{11}\sin^7\theta\cos^5\theta=20\sin 2\theta-5\sin 4\theta-10\sin 6\theta$$
$$+4\sin 8\theta+2\sin 10\theta-\sin 12\theta.$$

8. If
$$S \equiv x^2 + y^2 + 2gx + 2fy - 2fg = 0,$$
$$S' \equiv x^2 + y^2 + 2g'x + 2f'y - 2f'g' = 0$$
be the equations to two of a set of coaxial circles, shew that the point circles are given by
$$S^2(f'+g')^2 - 2SS'(f+g')(f'+g) + S'^2(f+g)^2 = 0.$$

9. A parabola touches the line $y = x$ at the origin, and its axis is parallel to the axis of x. The parabola also passes through the point $(4, -3)$. Find the equation to the parabola and the co-ordinates of its focus.

10. A piece of tin in the form of a quadrant of a circle is made into a hollow cone by bringing together the extreme radii. If this cone is suspended from a point of its rim, prove that in the position of equilibrium the axis will make an angle $\tan^{-1}\sqrt{\frac{3}{5}}$ with the vertical.

11. Two smooth inclined planes of masses M, M' and angles α, α' are placed edge to edge on a smooth horizontal table, and a rod of mass m is placed horizontally between them and perpendicular to the common edge. Shew that it will remain horizontal throughout its subsequent descent if
$$M'(M+m)\tan^2\alpha = M(M'+m)\tan^2\alpha'.$$

12. Prove the following properties of the curve
$$ax^3 - k(lx^2y + my^3) + hx + ky + c = 0,$$
viz.

(i) all its asymptotes pass through the origin,

(ii) six tangents can be drawn parallel to the axis of y,

(iii) their points of contact lie on the conic
$$lx^2 + 3my^2 = 1.$$

LVII.

1. Prove that the square on the side of a regular pentagon is equal to the sum of the squares on the sides of the regular hexagon and decagon, all inscribed in the same circle.

2. Construct an ellipse given one focus, one tangent, and one extremity of the minor axis.

3. Establish the identity

$$2\begin{vmatrix} a^2, & ac, & c^2 \\ a'^2, & a'c', & c'^2 \\ a''^2, & a''c'', & c''^2 \end{vmatrix}^2 = \begin{vmatrix} 0, & (ac'-a'c)^2, & (ac''-a''c)^2 \\ (ac'-a'c)^2, & 0, & (a'c''-a''c')^2 \\ (ac''-a''c)^2, & (a'c''-a''c')^2, & 0 \end{vmatrix}.$$

4. Sum the series

$$\frac{2^3}{1\,.\,3}+\frac{3^3}{2\,.\,4}\,.\,x+\frac{4^3}{3\,.\,5}\,.\,x^2+\frac{5^3}{4\,.\,6}\,.\,x^3+\ldots$$

to infinity, where $x<1$.

5. If $x_1, x_2 \ldots x_n$ are the roots of the equation

$$(a_1-\lambda)(a_2-\lambda)\ldots(a_n-\lambda)+k=0,$$

then $a_1, a_2 \ldots a_n$ are the roots of the equation

$$(x_1-\lambda)(x_2-\lambda)\ldots(x_n-\lambda)-k=0.$$

6. In a plane triangle ABC the square of the distance between the centres of the inscribed circle and the escribed circle which touches the side BC is $4R(r_1-r)$, and this distance subtends at the centre of the circumscribed circle an angle

$$\tan^{-1}\frac{2(\sin B-\sin C)}{2\cos A-1}.$$

7. If a regular polygon of 100 sides be inscribed in a circle, prove that its area differs from that of the circle by less than $\frac{1}{1500}$ of the latter.

8. TP, TQ are tangents to the parabola $y^2 = 4ax$ such that the normals at P, Q meet on the curve. Shew that the centre of the circle TPQ lies on the parabola

$$2y^2 = a(x - a).$$

9. If T be the pole of a chord PQ of a conic which subtends a constant angle 2α at the focus S, prove that

$$\frac{1}{SP} + \frac{1}{SQ} - \frac{2\cos\alpha}{ST}$$

is constant.

10. The weight of a common steelyard is Q, and the distance of the fulcrum from the point at which the body to be weighed hangs is a, when the instrument is in perfect adjustment. The fulcrum is displaced to a distance $a + \alpha$ from this end. Shew that the correction to be applied to give the true weight of a body which in the imperfect instrument appears to weigh W, is

$$(W + P + Q)\,\alpha/(a + \alpha),$$

where P is the moveable weight.

11. A hammer-head of mass m, moving with velocity v, is used to drive a nail of mass p horizontally into a block of mass M which is free to move on a smooth horizontal plane. Prove that the nail penetrates the block to a distance

$$\frac{Mm^2v^2}{2R(m+p)(M+m+p)},$$

where R is the resistance to penetration, supposed constant.

12. Prove that

$$\int_0^1 x^3 e^x \, dx = \cdot 56$$

correct to two decimal places.

LVIII.

1. Given three straight lines in space, no two of which are in the same plane, construct a parallelepiped having three of its edges along these straight lines.

2. Two plane sections of a cone have a common directrix: prove that the perpendiculars from the centres of the sections on the axis of the cone are to each other as the squares of the minor axes of the sections.

3. If n be a positive integer, then will

(i) $n^n (n+1)^{n+1} > (n+\frac{1}{2})^{2n+1}$,

(ii) $\left(n+\dfrac{1}{n}\right)^{2n} > (n-1)^{n-1}(n+1)^{n+1}$.

4. If $n+1$ and $2n+1$ are both prime numbers greater than 5, shew that n must be divisible by 6: also that either n, or $n-1$, or $n-3$ must be divisible by 5.

5. Prove that the equation

$$a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

will be satisfied by one of the nth roots of unity if

$$\begin{vmatrix} a_1 & a_2 & a_3 \ldots a_n \\ a_2 & a_3 & a_4 \ldots a_1 \\ a_3 & a_4 & a_5 \ldots a_2 \\ \cdots & \cdots & \cdots \\ a_n & a_1 & a_2 \ldots a_{n-1} \end{vmatrix} = 0.$$

6. A circle is described passing through the angular points B and C of a triangle ABC, and also through the middle point of AB. Prove that its area bears to that of the circumcircle of ABC the ratio

$$3 - 2\cos 2A - 2\cos 2B + \cos 2C : 4(1-\cos 2B).$$

7. Prove that

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \tfrac{3}{16},$$

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \tfrac{1}{16}.$$

8. If n be the length of a normal chord of an ellipse, and d the semi-diameter perpendicular to it, then

$$n = \frac{2d^3 ab}{(a^2 + b^2) d^2 - a^2 b^2}.$$

9. Shew that the envelope of a straight line which is such that the circles

$$x^2 + y^2 - 2kx + \delta^2 = 0, \quad x^2 + y^2 - 2k'x + \delta^2 = 0$$

make equal intercepts on it, is the parabola

$$y^2 - 2(k + k')x = 0.$$

10. A uniform ladder of length $2a$ rests inclined at an angle α to the vertical against a smooth rail at a height h above the ground. If λ be the angle of friction between the ladder and the ground, shew that a man of weight n times that of the ladder can ascend a distance along the ladder equal to

$$\frac{1}{n}\{2(n+1)h \sin\lambda \sec(\alpha - \lambda) \operatorname{cosec} 2\alpha - a\}$$

without the ladder slipping.

11. A particle which is performing a simple harmonic motion of period T about a centre O passes through a point P with velocity v in the direction OP. Prove that the time which elapses before its return to P is

$$\frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi . OP}\right).$$

12. Prove that $\log(\sec x + \tan x)$ is an odd function of x, and that its expansion in powers of x begins with

$$x + \frac{x^3}{3!} + \frac{x^5}{4!} + \frac{61x^7}{7!}.$$

LIX.

1. Through a point A within a circle are drawn chords PP', QQ'. Shew that the chords PQ, $P'Q'$ subtend equal angles at B, the point inverse to A with respect to the circle.

2. If P be any point of an ellipse whose foci are S and S', shew that the loci of the centres of two of the circles touching the sides of the triangle SPS' are straight lines, and the loci of the centres of the other two are ellipses.

3. Shew that the equations

$$ayz + b(y^2 + z^2) + (c + yz)(y + z) = 0,$$
$$azx + b(z^2 + x^2) + (c + zx)(z + x) = 0,$$
$$axy + b(x^2 + y^2) + (c + xy)(x + y) = 0,$$

admit of an infinite number of solutions if $ab = b^2 + c$, and that if this condition be not fulfilled, then two of the quantities x, y, z must be equal.

4. Prove that if n be a positive integer, then

$$2^n + \frac{n(n-1)}{1^2} \cdot 2^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1^2 \cdot 2^2} \cdot 2^{n-4} + \ldots = \frac{(2n)!}{(n!)^2}.$$

5. If one root of $x^3 + ax + b = 0$ is twice the difference of the other two, shew that the roots are

$$-\frac{13b}{12a}, \quad \frac{13b}{3a}, \quad -\frac{13b}{4a},$$

and that $144a^3 + 2197b^2 = 0$.

6. Two regular polygons of m and n sides are inscribed in two concentric circles of radii a and b respectively: prove that the sum of the squares on all the lines joining the vertices of the one to the vertices of the other is

$$mn(a^2 + b^2).$$

7. Prove that

$$\left(2\cos\frac{x}{2}-1\right)\left(2\cos\frac{x}{4}-1\right)\left(2\cos\frac{x}{8}-1\right)\ldots=\tfrac{1}{3}(2\cos x+1).$$

8. The centroid of a triangle inscribed in the hyperbola $xy=a^2$ is at the point $(ka, 0)$. Shew that its sides touch the conic

$$4xy=(a+3ky)^2.$$

9. Find the equations of the directrices of the conic

$$7x^2+7y^2-1+2y-2x+2xy=0,$$

and trace the curve.

10. Three equal uniform rods AD, BD, CD are hinged together at D. A is hinged to a point in a horizontal plane and B, C rest on this plane, symmetrically with respect to AD. Equilibrium is maintained by strings attached to B and C and to fixed points in the plane. Prove that the strings must pass through the circumcentre of the triangle ABC and that the tension of each is $3W\tan\alpha/8\cos^2\theta$, where W is the weight of a rod, α the angle the rods make with the vertical, and 2θ the angle BAC.

11. The line joining P to Q is inclined at an angle α to the horizontal. Shew that the least velocity required to shoot from P to Q is to the least velocity required to shoot from Q to P as

$$\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right):1.$$

12. Prove that the curve defined by

$$x=a\sin 2\theta,\quad y=b\sin\theta,$$

where θ is a variable parameter, consists of two loops, each of area $\frac{4}{3}ab$.

LX.

1. From a fixed point A straight lines are drawn to meet a fixed straight line in B and C, so that the rectangle $AB \,.\, AC$ is constant. Prove that the locus of the centre of the circle inscribed in the triangle ABC is a circle.

2. The major axes of two elliptic sections of a right circular cone intersect inside the cone but not on its axis, and the four extremities of these major axes lie on a circle. Prove that the sections are similar.

3. If r be an integer greater than unity, shew that

$$ab - (r+1)(a-1)(b-1) + \frac{(r+1)r}{2\,!}(a-2)(b-2)$$
$$- \frac{(r+1)r(r-1)}{3\,!}(a-3)(b-3) + \ldots = 0.$$

4. The arithmetic mean of a and b is u_1, that of b and u_1 is u_2, that of u_1 and u_2 is u_3 and so on. Prove that

$$u_n = \tfrac{1}{3}\left[\{1 - (-\tfrac{1}{2})^n\}\, a + \{2 + (-\tfrac{1}{2})^n\}\, b\right].$$

5. If α be an imaginary fifth root of unity, then the quantity $\alpha^2 + \alpha^4$ satisfies the equation

$$x^4 + 2x^3 + 4x^2 + 3x + 1 = 0.$$

6. Prove that if γ and δ be the two values of θ between 0 and π which satisfy the equation

$$\sin 2\theta \cos^2(\alpha + \beta) + \sin 2\alpha \cos^2(\beta + \theta) + \sin 2\beta \cos^2(\alpha + \theta) = 0,$$

then α and β will satisfy the equation

$$\sin 2\phi \cos^2(\gamma + \delta) + \sin 2\gamma \cos^2(\delta + \phi) + \sin 2\delta \cos^2(\gamma + \phi) = 0.$$

7. If n is a positive integer, prove that

$$2n^2 = \operatorname{cosec}^2 \frac{\pi}{4n} + \operatorname{cosec}^2 \frac{3\pi}{4n} + \ldots \text{ to } n \text{ terms.}$$

8. Tangents are drawn to the conic $ax^2 - by^2 = 1$ from any point on the conic $ax^2 + by^2 = 1$. Shew that the chord joining the other points in which these tangents meet the latter conic is a tangent to the conic

$$ax^2 + 9by^2 = 1.$$

9. Prove that the equation to the circle of curvature at the point $(1, -1)$ of the conic $3x^2 + 2xy + 3y^2 - 8x - 8y - 4 = 0$ is

$$3x^2 + 3y^2 - 11x - 9y - 4 = 0.$$

10. A smooth hemisphere is fixed on a rough horizontal plane, and a rod whose length is n times the radius of the hemisphere rests against the hemisphere in a vertical plane through the centre with its lower end on the horizontal plane. Prove that if λ be the angle of friction between the rod and the plane, the rod will rest in limiting equilibrium inclined to the horizon at an angle

$$\cot^{-1}\{\tfrac{1}{4}(n + \sqrt{n^2 + 8n\cot\lambda - 16})\},$$

provided this angle is real and $n\cot\lambda$ is not greater than $n^2 + 2$.

11. Particles are projected horizontally from different points in a tower of height h, each with a velocity due to the height of the tower above the point of projection. Prove that they will all cease to rebound from the horizontal plane through the foot of the tower within or on a circle of radius

$$\frac{1+e}{1-e} \cdot h$$

with its centre at the foot of the tower.

12. Shew that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n . n!}{x^{n+1}}\left\{\log x - \sum_1^n \left(\frac{1}{r}\right)\right\}.$$

LXI.

1. Describe a circle touching two given circles and also the line joining their centres.

2. If the normals at the points P and Q to a parabola meet in U, shew that the circumcircle of the triangle SPQ bisects TU.

3. Prove that if n is a positive integer the sum of the reciprocals of the coefficients in the expansion of $(1-x)^{2n}$ is

$$2-\frac{1}{n+1}.$$

4. Prove that

$$\frac{1^2 \cdot 2^2}{1!}+\frac{2^2 \cdot 3^2}{2!}+\frac{3^2 \cdot 4^2}{3!}+\frac{4^2 \cdot 5^2}{4!}+\ldots=27e.$$

5. Shew that the sum of the mth powers of the roots of the equation

$$x^n - x^{n-1} - x^{n-2} - \ldots - x^2 - x - 1 = 0$$

is $2^m - 1$ if $m \not> n$.

6. If G be the centre of gravity of a triangle ABC and l_1, l_2, l_3 the radii of the circles BGC, CGA, AGB, then

$$a^2 \cdot \frac{l_2 l_3}{l_1} + b^2 \cdot \frac{l_3 l_1}{l_2} + c^2 \cdot \frac{l_1 l_2}{l_3} = R\,(a^2+b^2+c^2).$$

7. If $\tan\theta = \frac{1}{2}$, find the numerical value of $\tan 7\theta$, and verify the result by means of a diagram drawn to scale.

8. Prove that the envelope of straight lines which are cut harmonically by the conics

$$ax^2 + by^2 = 1, \quad a'x^2 + b'y^2 = 1$$

is the conic

$$\frac{x^2}{b+b'} + \frac{y^2}{a+a'} = \frac{1}{ab'+a'b}.$$

9. If from a given point any two straight lines be drawn conjugate with regard to a given conic and meeting it in P, P', Q, Q', then the rectangular hyperbola through P, P', Q, Q' will pass through four fixed points.

10. An isosceles triangle ABC of weight W is moveable in a vertical plane about a hinge at the vertex A, and the point B is fastened to a string which passes over a pulley vertically above A at a distance from A equal to the perpendicular from A on BC, a weight P being suspended at the other end of the string. If the angle BAC is 2α and ϕ is the inclination of BC to the vertical, prove that

$$2W\cos\phi = 3P\cos(\alpha-\phi)/\{1-2\sin(\alpha-\phi)\cos\alpha+\cos^2\alpha\}^{\frac{1}{2}}.$$

11. During the motion of an Atwood's machine one of the masses receives a horizontal blow in the plane of the string which gives it a horizontal velocity due to falling freely through a height equal to n times the length of the string by which it hangs. Shew that the tension of the string is instantaneously increased in the ratio $n+1:1$.

12. Evaluate the indefinite integrals

$$\int\frac{dx}{x\sqrt{x^2-1}}, \qquad \int\frac{dx}{(a\cos x+b\sin x)^2},$$

and
$$\int\sin^3 x\cos^4 x\,dx.$$

LXII.

1. Shew that in general only one tetrahedron can be drawn which shall have two given finite straight lines for the shortest distances between two pairs of opposite edges, and give a construction for it.

2. Two parallel sides of a rectangle touch a given ellipse, and the other two sides touch a given confocal ellipse. Shew that the perimeter of the parallelogram formed by the points of contact of the tangents is constant.

3. Eliminate x, y, z from the equations

$$x^2(y+z)=a^3, \quad y^2(z+x)=b^3, \quad z^2(x+y)=c^3,$$
$$xyz=d^3.$$

4. If a, b and c are unequal, prove that it is impossible for the coefficients of two successive terms in the expansion of

$$\frac{1-ax}{(1-bx)(1-cx)}$$

to be both zero.

5. If
$$\phi(\lambda) \equiv \begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix},$$

express $\phi(\lambda)\,\phi(-\lambda)$ as a determinant and deduce that the roots of $\phi(\lambda)=0$ are all real.

6. If $\sin^2 A + \sin^2 B + \sin^2 C = 1$, prove that the circumcircle of the triangle ABC cuts the nine-point circle orthogonally.

7. Sum to infinity the series

(i) $\cos^3\theta - \frac{1}{3}\cos^3 3\theta + \frac{1}{3^2}\cos^3 3^2\theta - \frac{1}{3^3}\cos^3 3^3\theta + \ldots$

(ii) $\sin\theta\sin 2\theta + a\sin 2\theta\sin 3\theta + a^2\sin 3\theta\sin 4\theta + \ldots\ (a<1)$.

8. If any point P on the ellipse be joined to the extremities of the minor axis by straight lines meeting the minor auxiliary circle in Q, R, then QR touches the ellipse

$$x^2(a^2+b^2)^2+4a^2b^2y^2=4a^2b^4.$$

9. Shew that the common conjugate diameters of two concentric conics are harmonic conjugates with regard to the common diametral chords.

10. To a hollow hemispherical bowl a lid of the same substance and the same small uniform thickness is attached. The bowl is resting, with the plane of the lid inclined to the horizon, in limiting equilibrium against a rough vertical wall and an equally rough horizontal floor. Prove that the coefficient of friction must be less than

$$\frac{\sqrt{17}-3}{4}.$$

11. Particles, of equal mass m, are attached to the ends of an inextensible thread, and a particle, of mass nm, is attached at the middle. The system is on a smooth table, with the middle particle held at a point O, and the other two describing equal circles about it in the same sense and with the same speed, the angle between the two parts of the thread being α. Prove that if the particle at O is released, the tension of either part of the thread is suddenly diminished in the ratio

$$n : n+1+\cos\alpha.$$

12. Find the asymptotes of the curve $xy^3=a^3(a+x)$ and trace the curve, shewing that the point $(-a, 0)$ where the curve crosses the axis of x, is a point of inflexion.

LXIII.

1. A variable chord CD of a circle bisects a given chord AB in O. Shew that the locus of the intersection of AD and BC is a straight line parallel to AB.

2. An ellipse of given major axis has one focus at the focus of a given parabola and the ellipse touches the parabola. Shew that the ellipse always touches another fixed parabola.

3. From the Arithmetic Progression a, $a+b$, $a+2b\ldots$ are formed in succession the series s_1, $s_2 \ldots s_p$ which are such that the rth term of each is equal to the sum of the first r terms of the preceding series. Prove that the nth term of s_p is

$$\frac{(n+p-1)!}{(p+1)!\,(n-1)!}\{(p+1)\,a+(n-1)\,b\}.$$

4. If four odd and four even integers are written down at random in a row, the chance that no two odd integers are adjacent is $\frac{1}{14}$.

5. Prove that the equation

$$ax^5+bx^3+cx^2+d=0$$

will have three equal roots if

$$-\frac{c}{b}=\frac{b^2}{5ac}=\frac{5bd}{c^2},$$

each of these quantities being equal to the repeated root.

Solve the equation

$$32x^5-360x^3+540x^2-243=0.$$

6. Prove that

$$\Sigma\cos(\pm\alpha_1\pm\alpha_2\pm\alpha_3\pm\ldots\pm\alpha_n)=2^n\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n,$$

every possible variation of the signs being taken.

7. If $(1+x)\tan\alpha = (1-x)\tan\beta$ and $-1<x<1$, prove that

$$n\pi + \beta - \alpha = x\sin 2\beta - \tfrac{1}{2}x^2\sin 4\beta + \tfrac{1}{3}x^3\sin 6\beta - \ldots,$$

where n is some integer.

8. If two concentric ellipses touch one another, the angle between their major axes is

$$\tan^{-1}\sqrt{\frac{(a'^2-a^2)(b'^2-b^2)}{(a'^2-b^2)(a^2-b'^2)}},$$

the semi-axes of the two ellipses being a, b and a', b'.

9. Prove that the normals at the points α, β, γ of the parabola

$$\frac{2a}{r} = 1 + \cos\theta$$

are concurrent if

$$\tan\tfrac{1}{2}\alpha + \tan\tfrac{1}{2}\beta + \tan\tfrac{1}{2}\gamma = 0.$$

10. A rough stick rests on the rim of a flower-pot, whose shape is that of a truncated cone of semi-vertical angle α. Its lower end is in contact with a slant side, the vertical plane through the stick passing through the axis of the cone. Prove that if the stick, when on the point of slipping down, is inclined to the horizontal at an angle β, then its length is

$$2c\cos\alpha\cos\lambda\cos(\alpha+\lambda)\sec\beta\sec(\beta-\alpha)\sec(\beta-\alpha-2\lambda),$$

where λ is the angle of friction, and c the diameter of the pot at the top.

11. Two bodies of equal mass impinge directly: shew that if e is the coefficient of restitution, the amount of kinetic energy apparently gained by the one body will be e times the amount apparently lost by the other, provided that before impact the bodies are moving in opposite directions with speeds proportional to $3+e$ and $1-e$.

12. Prove that the mean value of the ordinate of a semi-ellipse (supposing the ordinates to be taken equidistant) is the length of a semi-quadrant of the minor auxiliary circle.

LXIV.

1. Three concurrent straight lines through the vertices A, B, C of a triangle meet the opposite sides in P, Q, R respectively and R' is the point dividing BA externally in the same ratio as R divides it internally. Prove that the middle points of AP, BQ, CR' are collinear.

2. Through Q, a point of intersection of two confocal and coaxial parabolas, a focal chord $QP'P$ is drawn terminated by them. Shew that

$$SP \cdot SP' = SQ^2.$$

3. If $a_0, a_1, a_2 \ldots a_n$ are the successive terms of an Arithmetic progression of which the common difference is d, and if c_r is the coefficient of x^r in the expansion of $(1+x)^n$, prove that

$$c_0 a_0^2 + c_1 a_1^2 + c_2 a_2^2 + \ldots + c_n a_n^2 = 2^{n-2}\{(a_0 + a_n)^2 + nd^2\}.$$

4. If
$$p_0 + p_1 x + p_2 x^2 + \ldots$$
and
$$q_0 + q_1 x^{-1} + q_2 x^{-2} + \ldots$$
be possible expansions of $\dfrac{1}{x^2 + ax + b}$ for different values of x prove that

$$q_{n+1} = b^n p_{n-1}.$$

5. If α, β, γ be the roots of the cubic $x^3 + ax + b = 0$, then the roots of the quadratic

$$x^2 - abx + a^5 + 7a^2b^2 = 0$$

are
$$\alpha^3\beta^2 + \beta^3\gamma^2 + \gamma^3\alpha^2 \text{ and } \alpha^3\gamma^2 + \beta^3\alpha^2 + \gamma^3\beta^2.$$

6. If l, m, n are the perpendiculars from the angles of a triangle on the line joining the incentre and the circumcentre, prove that

$$l : m : n = r_2 r_3 (r_2 - r_3)/(r_2 + r_3) : r_3 r_1 (r_3 - r_1)/(r_3 + r_1) : r_1 r_2 (r_1 - r_2)/(r_1 + r_2).$$

7. Prove that $\cos^{\frac{1}{n}} x$ can be expressed by

$$\left(1-\frac{2n+3}{12n}\,.\,x^2\right) \div \left(1-\frac{2n-3}{12n}\,.\,x^2\right)$$

accurately as far as x^4, and that the error in the coefficient of x^6 is

$$\frac{4n^2-5}{480n^3}.$$

8. Two ellipses have coincident axes and a common tangent is drawn. Shew that the length intercepted between the points of contact is to the length intercepted by the axes as

$$(a^2-a'^2)(b'^2-b^2) : a^2b'^2-a'^2b^2,$$

where a, b and a', b' are the respective semi-axes.

9. Prove that the normals to the rectangular hyperbola $xy=c^2$ at the extremities of the chord $px+qy=1$ intersect in the point

$$\frac{1}{p}-c^2\left(q-\frac{p^2}{q}\right),\ \frac{1}{q}-c^2\left(p-\frac{q^2}{p}\right).$$

10. Four equal weightless rods are hinged together so as to form a rhombus $ABCD$: equal weights are attached to the four corners and the corner A is attached to a point on a rough inclined plane whose inclination α to the horizon is greater than λ the angle of friction. If the weights rest on the plane, while the rods are just clear of it, prove that the greatest angle the rods can make with the line of greatest slope is

$$\sin^{-1}\frac{\tan\lambda}{2\tan\alpha-\tan\lambda}.$$

11. A particle is projected inside a tube in the form of a parabola whose axis is vertical and vertex upwards. Prove that in the subsequent motion the pressure on the tube at any point is proportional to the curvature.

12. P, Q are two points on a circle whose centre is C and radius a. Prove that the maximum value of the radius of the circle inscribed in the triangle CPQ is

$$\frac{a}{2}\sqrt{10\sqrt{5}-22}.$$

LXV.

1. On the base BC of a triangle ABC find a point D such that the rectangle $BD \,.\, DC$ together with the square on AD may be equal to a given square.

2. Given a focus and two tangents to an ellipse prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

3. If each x and each y be either zero or positive or negative unity, the number of different solutions of the equation

$$x_1 x_2 \dots x_n = y_1 y_2 \dots y_n$$

is $3\,(3^{2n-1} - 3^{n-1} \,.\, 2^{n+1} + 2^{2n-1})$.

4. Sum the series

$$1 + \frac{2}{1\,!} + \frac{3}{2\,!} + \frac{8}{3\,!} + \frac{13}{4\,!} + \frac{30}{5\,!} + \frac{55}{6\,!} + \dots \text{ to infinity.}$$

5. If the equation $x^4 + 6ax^2 + 4bx + c = 0$ can be written in the form

$$m\,(x-n)^4 - n\,(x-m)^4 = 0,$$

shew that $a^3 + b^2 = ac$, and that m and n are the roots of the equation

$$at^2 + bt - a^2 = 0.$$

Solve the equation

$$x^4 + 12x^2 + 8x + 6 = 0.$$

6. Shew that the triangle whose vertices are the incentre, circumcentre, and orthocentre of a given triangle is always obtuse-angled.

7. Prove that

$$\sin\theta\cos^2\theta = \theta - \frac{7}{3\,!}\theta^3 + \frac{61}{5\,!}\theta^5 - \ldots,$$

and find the general term of the series.

8. Two equal rods AB, BC each of length a are rigidly jointed at B so as to include an angle 2α. If A and B slide along two rectangular axes, prove that C will trace an ellipse of area

$$2\pi a^2 \sin^2\alpha.$$

9. A conic has four-point contact with the parabola $y^2 = 4ax$ and the radius of its director circle is constant and equal to c. Prove that the locus of its centre is the curve

$$(y^2 - 4ax)(y^2 + 4ax + 8a^2) + 16a^2c^2 = 0.$$

10. AB and BC are two uniform rods of equal lengths, but of unequal weights W and W' respectively. They are inclined to the horizontal at an angle α and jointed at C, the ends A and B being fastened to two fixed points in the same horizontal. Prove that the horizontal and vertical components of the reaction at C are respectively

$$\tfrac{1}{4}(W + W')\cot\alpha \text{ and } \tfrac{1}{4}(W - W').$$

11. A shell moving in any manner bursts into three pieces of masses m_1, m_2, m_3. Prove that the increase of kinetic energy due to the explosion is

$$\frac{1}{2}\,\frac{m_2m_3{v_{23}}^2 + m_3m_1{v_{31}}^2 + m_1m_2{v_{12}}^2}{m_1 + m_2 + m_3},$$

where v_{pq} is the relative velocity of m_p and m_q after the explosion.

12. If $x\,.\,\psi(a) + y\,.\,\phi(a) = 1$ be the equation of a tangent to a curve, shew that the equation to the corresponding normal is

$$\{x\,.\,\phi(a) - y\,.\,\psi(a)\}\{\psi(a)\,\phi'(a) - \psi'(a)\,\phi(a)\} = \phi(a)\,\phi'(a) + \psi(a)\,\psi'(a).$$

LXVI.

1. Through the orthocentre P of a triangle ABC is drawn a normal to its plane and on this line a point D is taken. Prove that the perpendiculars from the vertices A, B, C to the opposite faces of the tetrahedron $ABCD$ intersect PD in the same point.

2. AB, BC, CD, DA are four tangents to a conic and P, Q, R, S are their respective points of contact. Shew that AC, BD, PR, QS are concurrent.

3. The sum of the homogeneous products of r dimensions of the numbers 1, 2, 3 ... n is

$$\sum_{p=1}^{p=n} \frac{(-1)^{n-p} \cdot p^{n+r-1}}{(p-1)!\,(n-p)!}.$$

4. If p be prime, then

$$\{1 \cdot 3 \cdot 5 \cdot 7 \ldots (p-2)\}^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.$$

5. If α, β, γ be the roots of $x^3 + 3px + q = 0$, the equation whose roots are $\alpha(\beta - \gamma)$ and two similar expressions is

$$x^3 = 9p^2x + 3q\sqrt{-3(q^2 + 4p^3)}.$$

6. A quadrilateral whose sides are a, b, c, d is such that a circle can be inscribed in it and another described about it. Prove that the tangents from the angular points to the inscribed circle are ab/s, bc/s, cd/s, da/s, where $2s = a + b + c + d$.

7. If $\alpha = \dfrac{2\pi}{17}$, prove that

$$\sum_{n=0}^{n=7} \cos(2n+1)\alpha = -\tfrac{1}{2},$$

$$\sum_{n=0}^{n=7} \cos^2(2n+1)\alpha = 3\tfrac{3}{4}.$$

8. Shew that the common tangent of the conics

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha), \quad \frac{l'}{r} = 1 + e'\cos(\theta - \alpha'),$$

subtends a right angle at the focus if

$$l^2(1 - e'^2) + l'^2(1 - e^2) + 2ll'ee'\cos(\alpha - \alpha') = 0.$$

9. A chord of an ellipse, whose semi-axes are a, b, subtends angles $2\theta_1$, $2\theta_2$ at the foci, and touches a similar, similarly situated and concentric ellipse, whose semi-axes are $a\cos\beta$, $b\cos\beta$. Prove that

$$b(\cot\theta_1 + \cot\theta_2) = 2a\cot\beta.$$

10. Two equal uniform rods, each of weight W and length a, are freely jointed at one extremity, their other ends being connected by a string of length $2c$. The rods are placed symmetrically over a smooth horizontal cylinder of radius b. Prove that the tension of the string is

$$W\left(\frac{ab}{c^2} - \frac{c}{2\sqrt{a^2 - c^2}}\right).$$

11. From a point in a fixed horizontal plane a ball A is projected upwards with velocity V and when it is at its highest point another ball B is thrown upwards from the horizontal plane with the same velocity, so that the balls constantly pass each other and rebound from the fixed plane. Shew that $2n + 1$ impacts of the balls take place in alternate order, where n is the integer next below

$$\log 2/\log(e^{-1}),$$

and that at the rth impact of B the velocity of A and at the $(r+1)$th impact of A the velocity of B is $(1 - e^r)V$, downwards and upwards respectively.

12. The ordinate NP of any point of an ellipse of semi-axes a and b is produced to Q, so that $NP \cdot NQ = c^2$, c being a constant greater than b. Prove that the area included between the locus of Q, the ellipse and the tangents at the ends of its major axis, is

$$\frac{a(2c^2 - b^2)}{b}\pi.$$

LXVII.

1. In a given triangle ABC inscribe another triangle having its sides of given lengths.

2. Prove that, if the axes of two parabolas be at right angles, the circle round the triangle formed by the three common tangents at a finite distance is the circle on the line joining the foci as diameter.

3. If $C_m = (4n-2m)!/\{(2n-2m)!\,(2n-m)!\,m!\}$, prove that
$$C_0 - C_1 + C_2 - C_3 \ldots + (-1)^n C_n = 4^n.$$

4. If the quantities involved be all positive, prove that $(bcd+cda+dab+abc)^{a+b+c+d}$ is not less than
$$a^{b+c+d-a} \,.\, b^{c+d+a-b} \,.\, c^{d+a+b-c} \,.\, d^{a+b+c-d} \,.\, (a+b+c+d)^{a+b+c+d}.$$

5. Prove that if x_1, x_2, x_3, the roots of the equation
$$x^3 + ax + b = 0,$$
are connected by the relation $x_1 + 2x_2 + 3x_3 = k$, then
$$x_1 + \tfrac{1}{2}k = -\tfrac{1}{2}x_2 = x_3 - \tfrac{1}{2}k = -\frac{3b}{2(a+k^2)}.$$

6. Prove that if
$$\frac{\cos y \cos z}{\cos^2 a} + \frac{\sin y \sin z}{\sin^2 a} = 1,$$
$$\frac{\cos z \cos x}{\cos^2 a} + \frac{\sin z \sin x}{\sin^2 a} = 1,$$
then either $$\frac{\cos x \cos y}{\cos^2 a} + \frac{\sin x \sin y}{\sin^2 a} + 1 = 0,$$
or else $x - y = 2n\pi$.

7. A regular polygon of n sides (n being odd) is inscribed in a circle of radius a and the lines joining O, a point outside the circle distant c from its centre, to the corners of the polygon meet the circle again in $B_1, B_2 \ldots B_n$. If one of these lines passes through the centre, prove that
$$OB_1 \,.\, OB_2 \ldots OB_n = (c^2 - a^2)^n/(c^n \pm a^n),$$
and explain when the positive sign is to be taken in the denominator.

8. A triangle is self-conjugate with respect to a parabola. Prove that the perpendiculars to its sides at their points of intersection with the axis of the parabola are concurrent.

9. Prove that the square of the radius of the circle which can be drawn through the points of intersection of the conics

$$\alpha x^2 + \beta y^2 = 1, \quad ax^2 + by^2 + 2gx + 2fy + c = 0$$

is

$$\frac{(a-\beta)^2 (g^2 + f^2)}{(a\beta - b\alpha)^2} + \frac{(\alpha - \beta) c + (a - b)}{a\beta - b\alpha},$$

and that the square of the real semi-axis of the rectangular hyperbola through the same four points is

$$\frac{(\alpha + \beta)^2 (g^2 - f^2)}{(a\beta - b\alpha)^2} - \frac{(\alpha + \beta) c + (a + b)}{a\beta - b\alpha}.$$

10. Two inclined planes make angles α and β with the horizon in opposite directions and a rough uniform cylinder of weight W and radius a rests with its curved surface on them, and with its axis horizontal and parallel to the intersection of the planes: shew that the least couple which will move the cylinder is

$$Wa \sin \epsilon \{\sin (\beta + \epsilon) + \sin (\alpha - \epsilon)\} / \sin (\alpha + \beta),$$

where ϵ is the angle of friction and $\epsilon < \alpha < \beta$. What is the result if $\epsilon > \alpha < \beta$?

11. Two weights P and Q $(P > Q)$ are attached to the ends of a string which hangs over a smooth pulley. A third weight R $(Q + R > P)$ rests on an inelastic plane vertically below Q, being attached to Q by a string. If P be allowed to descend, and the system be left to itself, prove that it will come to rest after a time

$$\frac{4V}{g} \cdot \frac{P(P+Q)}{(P-Q)(R+Q-P)},$$

from the instant of the first tightening of the string, V being the velocity just before that instant.

12. Prove that

$$\frac{d^n}{dx^n}\left\{\frac{1}{x(x^2+1)}\right\} = (-1)^n n! \left[x^{-n-1} - \cos \{(n+1) \cot^{-1} x\} (x^2+1)^{-\frac{n+1}{2}}\right].$$

LXVIII.

1. Points E and F are taken on the sides AC, AB respectively of the triangle ABC such that $AE = p \,.\, AC$, $AF = q \,.\, AB$ and BE, CF intersect in P. Shew that the areas of the triangles FPE, ABC are in the ratio

$$pq\,(1-p)\,(1-q) \;:\; 1-pq.$$

2. Four tangents to an ellipse (axes $2a$ and $2b$) form a parallelogram of area Δ. Prove that their points of contact form a parallelogram of area $8a^2b^2/\Delta$.

3. Prove that if u, v are the values of y for which the quadratic in x

$$\frac{(x-\alpha)(x-\beta)}{(x-\gamma)(x-\delta)} = \frac{(y-\alpha)(y-\beta)}{(y-\gamma)(y-\delta)}$$

has equal roots, then

$$\frac{(u-\alpha)(u-\gamma)}{(u-\beta)(u-\delta)} = \frac{(v-\alpha)(v-\gamma)}{(v-\beta)(v-\delta)} \text{ and } \frac{(u-\alpha)(u-\delta)}{(u-\beta)(u-\gamma)} = \frac{(v-\alpha)(v-\delta)}{(v-\beta)(v-\gamma)}.$$

4. In a certain examination there are three papers, for each of which 200 marks is the maximum. If less than 50 marks be obtained in a paper, they are not counted towards the aggregate. In order to pass a candidate must get at least 200. If a candidate just passes, shew that there are 1632 ways in which he could have secured his marks.

5. If x_1, x_2, x_3, x_4 be the roots of the equation

$$x^4 + px^3 + q = 0,$$

find the equation whose roots are $\dfrac{x_1 + x_2}{x_3 + x_4}$ and all similar quantities.

6. Shew that the equation $\sec x = x$ has a root approximately equal to

$$a - \frac{1}{a} - \frac{7}{6a^3},$$

where $a = (2n + \frac{1}{2})\,\pi$, and n is a large positive integer.

7. Prove that

$$1 + \frac{2}{1!} a \cos \alpha + \frac{3}{2!} a^2 \cos 2\alpha + \dots + \frac{n+1}{n!} a^n \cos n\alpha + \dots$$
$$= e^{a \cos \alpha} [\cos (a \sin \alpha) + a \cos (\alpha + a \sin \alpha)].$$

8. At each point of a parabola is drawn the rectangular hyperbola of four-point contact. Prove that the locus of the centre is the reflection of the parabola in its directrix.

9. Prove that if x, y, the co-ordinates of a point on a conic, are given in terms of the variable parameter t by the equations

$$x = \frac{at^2 + 2bt + c}{At^2 + 2Bt + C}, \quad y = \frac{a't^2 + 2b't + c'}{At^2 + 2Bt + C},$$

the area of the curve is

$$\frac{\pi}{2(AC - B^2)^{\frac{3}{2}}} \begin{vmatrix} a & b & c \\ a' & b' & c' \\ A & B & C \end{vmatrix}.$$

10. A rhomboidal framework $ABCD$ is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Shew that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude

$$W(2a^2c - b^3)/b^2(4a^2 - b^2)^{\frac{1}{2}}.$$

11. Shew that the least velocity with which a stone can be thrown so as to clear two points A and B at heights h_1 and h_2 is $\sqrt{g(h_1 + h_2 + d)}$, where d is the length AB.

12. If $$u_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx,$$

prove that

$$u_n + n(n-1) u_{n-2} = n(\tfrac{1}{2}\pi)^{n-1},$$

and hence that

$$u_3 = \tfrac{3}{4}\pi^2 - 6.$$

LXIX.

1. From a given point in the plane of a parallelogram draw a straight line which shall divide the parallelogram into two parts whose areas are in a given ratio.

2. P is a point on a rectangular hyperbola of which QR is a diameter. Prove that the tangents at P to the hyperbola and to the circle PQR form with PQ and PR a harmonic pencil.

3. Prove that if

$$x^2 = (\beta - \gamma)^2 + (\beta' - \gamma')^2, \quad y^2 = (\gamma - \alpha)^2 + (\gamma' - \alpha')^2,$$
$$z^2 = (\alpha - \beta)^2 + (\alpha' - \beta')^2,$$

then

$$4 \begin{vmatrix} 1 & \alpha & \alpha' \\ 1 & \beta & \beta' \\ 1 & \gamma & \gamma' \end{vmatrix}^2 = (x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

4. If $\quad u_n = n^r - n(n-1)^r + \dfrac{n(n-1)}{2!}(n-2)^r - \dots,$

prove that

$$\frac{u_1}{1} - \frac{u_2}{2} + \frac{u_3}{3} - \dots \pm \frac{u_r}{r} = 0,$$

if r be a positive integer greater than unity.

5. Shew that the roots of

$$x^3 - \tfrac{1}{3}p^2x - \tfrac{2}{27}p^3 - q^2 = 0$$

differ by a constant quantity from the squares of the corresponding roots of

$$x^3 + px + q = 0.$$

6. Shew that $\dfrac{\pi}{6}$ is the only real angle between 0 and π which satisfies the equation

$$2\cos\theta - \sqrt{3}\sin\theta = 2\sin\theta\cos\theta.$$

7. Prove that the infinite product

$$\left(1-\tan^2\frac{x}{2}\right)\left(1-\tan^2\frac{x}{2^2}\right)\left(1-\tan^2\frac{x}{2^3}\right)\ldots$$

converges to the value $x\cot x$.

8. If F be the fixed point on the normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ through which pass all chords subtending a right angle at P, and if ρ be the radius of curvature at P, prove that

$$PF=2\frac{a^{\frac{4}{3}}b^{\frac{4}{3}}\rho^{\frac{1}{3}}}{a^2+b^2},$$

and deduce the corresponding expression for the parabola.

9. Conics are drawn so as to pass through three given points not in one straight line, and to touch a given straight line not passing through any of the points. Shew that their centres lie on a curve of the fourth degree.

10. $ABCD$, $A'B'C'D'$ are any two positions of a rectangle in the same plane, G and G' the middle points of the diagonals. Prove that forces represented by AA', BB', CC', DD' are equivalent to a force represented by $4GG'$ and a couple whose moment is represented by $AC^2\sin\theta$, where θ is the angle between AC and $A'C'$.

11. A smooth wedge of angle α and mass M rests upon a horizontal table and a small inelastic spherical ball of mass m falls vertically upon it. Prove that the ball will rebound in a direction inclined to the face at an angle

$$\tan^{-1}\{(Me-m\sin^2\alpha)/(M+m\sin^2\alpha)\tan\alpha\},$$

where e is the coefficient of restitution between the ball and the wedge, which is not supposed to be tilted by the impact.

12. Prove that the area included between the curves

$$x^3+y^3=3axy \quad\text{and}\quad y^2=4ax$$

is $\frac{1}{6}a^2$.

LXX.

1. In a triangle ABC, AL, BM, CN are the perpendiculars on the sides and MN, NL, LM produced meet the sides in P, Q, R respectively. Shew that the circumcentres of the triangles ALP, BMQ and CNR are collinear.

2. If SP, $S'Q$ be parallel focal distances of an ellipse towards the same parts, the tangents at P and Q will intersect on the auxiliary circle.

3. If none of the quantities x, y, z be zero, the result of eliminating them from the equations

$$(x+y)(x+z) = bcyz, \quad (y+z)(y+x) = cazx,$$
$$(z+x)(z+y) = abxy,$$

is

$$\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix} = 0.$$

4. A, B and C have n articles to divide between them subject to the condition that B must not have more articles than either A or C. Shew that the number of arrangements is the coefficient of x^n in the expansion of $(1-x)^{-2}(1-x^3)^{-1}$.

5. Shew that the equation

$$x^4 + 4ax^3 + 6(a^2+b^2)x^2 + 4cx + d = 0$$

cannot have more than two real roots.

6. ABC is a triangle, N the centre of its nine-point circle and P any point. Prove that

$$\Sigma PA^2(1+\cot B\cot C) = R^2(3 + 8\cos A\cos B\cos C) + 4PN^2.$$

7. If $\quad y = x - m\sin 2x + \dfrac{m^2}{2}\sin 4x - \dfrac{m^3}{3}\sin 6x + \ldots,$

where
$$m = \tan^2\frac{\phi}{2},$$

shew that
$$\tan y = \cos\phi\tan x.$$

8. Prove that

$$\frac{r}{l}(1+e\cos\alpha)^2=\cos(\theta-\alpha)+e\cos(\theta-2\alpha)$$

is a circle passing through the origin and touching the conic

$$\frac{l}{r}=1+e\cos\theta.$$

9. A parabola circumscribes a triangle ABC and has its focus at the orthocentre. Prove that

$$\frac{\cos\frac{1}{2}A}{\sqrt{\cos A}}+\frac{\cos\frac{1}{2}B}{\sqrt{\cos B}}+\frac{\cos\frac{1}{2}C}{\sqrt{\cos C}}=0.$$

10. Two rough spheres each upon one of the faces of a double inclined plane are in equilibrium with a horizontal board resting on them, which is not in contact with the ridge of the plane. The figure is in all respects symmetrical. Prove that the coefficient of friction between the board and a sphere is not less than $\left(1+\frac{1}{n}\right)\tan\frac{1}{2}\alpha$, where α is the angle of either face and $2n$ the ratio of the weight of the board to that of either sphere.

11. An elastic particle is projected from a point H in one vertical wall, so as to strike a second parallel wall at a distance c and after rebounding strike the first wall again at K. Prove that for a given velocity of projection $\sqrt{2gh}$, the greatest possible height of K above H is

$$h-\frac{1}{4}\frac{(1+e)^2c^2}{e^2h}.$$

12. Prove that at any point of a rectangular hyperbola

$$3\rho\frac{d^2\rho}{ds^2}-2\left(\frac{d\rho}{ds}\right)^2-18=0.$$

LXXI.

1. D, E, F are the points on the circumcircle of a triangle whose pedal lines pass through the nine-point centre. Prove that the triangle DEF is equilateral.

2. The focus of the parabola touching the four sides of a cyclic quadrilateral lies on the third diagonal of the quadrilateral.

3. Find in the scale of 7 a number of four digits which is halved if the last digit is put in front of the other three.

4. If p be prime, and

$$(1+x)^{p-2} = 1 + a_1x + a_2x^2 + \dots,$$

then $a_1 + 2$, $a_2 - 3$, $a_3 + 4$ ... etc. are all multiples of p.

5. If the quantities A_1, A_2, A_3 ... A_n be all positive, then the roots of the equation

$$\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n} = 0$$

are all real.

6. If O be a point on the circumcircle of a triangle ABC, and if OA, OB, OC meet BC, CA, AB in P, Q, R respectively, prove that (with a convention as regards sign)

$$\frac{\cos A}{AP} + \frac{\cos B}{BQ} + \frac{\cos C}{CR} = 0.$$

7. Prove that the equation whose roots are

$$\cos^2 \frac{r\pi}{9} \quad (r = 1, 2, 3, 4)$$

is

$$256x^4 - 448x^3 + 240x^2 - 40x + 1 = 0.$$

8. TP, TQ are two tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If the centre of the circle TPQ be on the line $a^2x-b^2y=0$, prove that the locus of T is

$$(x-y)(x^2+y^2)+(a^2-b^2)(x+y)=0.$$

9. Two parabolas are drawn having three-point contact with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point whose eccentric angle is α and passing through the point β. Shew that their axes are inclined at an angle ϕ where

$$\cot\phi=\frac{a^2-b^2}{2ab}\sin\frac{3\alpha+\beta}{2}.$$

10. A sphere rests on three smooth pegs which lie in a horizontal plane and are at distances a, b, c from one another. Prove that the pressures on the pegs are in the ratio

$$a^2(b^2+c^2-a^2):b^2(c^2+a^2-b^2):c^2(a^2+b^2-c^2).$$

11. A train starts from rest at one station and stops at the next, the pull of the engine having one constant value when the train is getting up speed, and another when it is running at full speed. Prove that the work done by the engine in getting up speed exceeds that done by the brakes in stopping the train by $\left(\frac{V}{v}-1\right)$ times the work done by the resistance of the rails during the journey, V being the full speed, and v the average speed of the train.

12. Prove that the area of the loop of the curve

$$r\cos\theta=a\cos 2\theta$$

is

$$\tfrac{1}{2}(4-\pi)a^2,$$

and that the volume generated by the revolution of the loop about the initial line is

$$2\left(\log_e 2-\tfrac{2}{3}\right)\pi a^3.$$

LXXII.

1. Given three points A, B, C and two parallel straight lines, find a point P on one of the lines, such that if PA, PB, PC cut the other line in α, β, γ, then shall $\alpha\beta = \beta\gamma$.

2. A line moves so that the sum of the squares of the perpendiculars on it from two given points is constant. Shew that its envelope is a conic whose real foci, together with the two given points, form a square.

3. Prove the formula

$$\frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_n}$$

$$\equiv u_1 u_2 \dots u_n \left\{ \frac{1}{(u_2 - u_1)(u_3 - u_1)\dots(u_n - u_1)u_1^2} + \frac{1}{(u_1 - u_2)(u_3 - u_2)\dots(u_n - u_2)u_2^2} + \frac{1}{(u_1 - u_3)(u_2 - u_3)\dots(u_n - u_3)u_3^2} + \dots \right\}.$$

4. If m be any positive quantity greater than unity, prove that
$$m^{-1} + \sqrt[m]{m} < 2,$$
where $\sqrt[m]{m}$ has its real positive value.

5. If the sides of a triangle are the roots of the equation
$$x^3 - px^2 + qx - r = 0,$$
the triangle will be acute-, right-, or obtuse-angled according as
$$p^6 + 8p^2q^2 + 8p^3r + 8r^2 \overset{<}{\underset{>}{=}} 6p^4q + 16pqr.$$

Prove also that the radius of the circumcircle is
$$\frac{r}{\sqrt{4p^2q - 8pr - p^4}}.$$

6. If α, β, γ be unequal angles such that
$$k \cos \alpha \cos \tfrac{1}{2}(\beta - \gamma) = \cos \tfrac{1}{2}(\beta + \gamma),$$
$$k \cos \beta \cos \tfrac{1}{2}(\gamma - \alpha) = \cos \tfrac{1}{2}(\gamma + \alpha),$$

then will $$k \cos\gamma \cos\tfrac{1}{2}(\alpha-\beta) = \cos\tfrac{1}{2}(\alpha+\beta).$$
Prove also that $\Sigma \sin(\beta+\gamma)=0$, and

$$\left(1+\frac{1}{k}\right)\sin\tfrac{1}{2}(\alpha+\beta+\gamma) = -2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma.$$

7. If $[\sin(\alpha+i\beta)]^{p+iq} = A+iB$, prove that

$$\tan^{-1}\frac{B}{A} = \tfrac{1}{2}q\log(\sin^2\alpha+\sinh^2\beta) + p\tan^{-1}(\tanh\beta\cot\alpha).$$

8. If P be any point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and Q the centre of curvature at P of the confocal hyperbola through P, shew that the locus of Q is the curve

$$a^6y^2+b^6x^2 = (a^2-b^2)^2x^2y^2.$$

9. Prove that the centres of the circles which touch the sides of a triangle lie on any rectangular hyperbola with respect to which the given triangle is self-conjugate.

10. A fine wire ring of radius r and weight W is placed on a smooth fixed horizontal pole with its cross section of radius a. The plane of the ring is vertical and makes an angle θ with the planes perpendicular to the axis of the pole. The ring rests on the pole in contact with it at two points only, both on the same level, and equilibrium is preserved by a couple with its plane horizontal, acting on the wire. Prove that the moment of the couple is

$$W\sec^2\theta\,\frac{a^2-r^2\cos^4\theta}{\sqrt{r^2\cos^2\theta-a^2}}.$$

11. A particle is projected from a point on the inner circumference of a circular hoop, free to move in a horizontal plane. Prove that if the particle return to the same point of the hoop after three impacts, two of the impacts take place at the ends of a diameter of the hoop.

12. Find the asymptotes of the cubic

$$ax^2y+bxy^2+a'x^2+b'y^2+a''x+b''y=0$$

and shew that the three points in which the asymptotes meet the curve lie on the straight line

$$b^2(a^2a''+a'^2b)\,x + a^2(b^2b''+ab'^2)\,y + a'b'(a^2b'+a'b^2)=0.$$

LXXIII.

1. ABC is a triangle inscribed in a circle and the tangents to the circle at A, B, C meet BC, CA, AB in A', B', C' respectively. Shew that the middle points of AA', BB', CC' lie on the radical axis of the circumcircle and the nine-point circle.

2. If a parabola touches the sides of a given triangle, each of the chords of contact will pass through a fixed point.

3. Shew that the sum of all numbers of three different digits, not beginning with a cypher, is 355680.

4. Shew that the value of the infinite continued fraction

$$\frac{1}{a-}\,\frac{a}{2a+1-}\,\frac{2a}{3a+1-}\,\frac{3a}{4a+1-}\ldots$$

is $e^{\frac{1}{a}}-1$.

5. If s_r denotes the sum of the rth powers of the reciprocals of the roots of the equation

$$1+\frac{x}{1\,!}+\frac{x^2}{2\,!}+\ldots+\frac{x^n}{n\,!}=0,$$

shew that

$$s_2=s_3=\ldots=s_n=0, \quad s_{n+1}=\frac{1}{n\,!}.$$

6. From the equations

$$x\sin\theta-y\cos\theta=-\sin 4\theta,$$

$$2(x\cos\theta+y\sin\theta)=5-3\cos 4\theta,$$

deduce that

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2.$$

7. If n be an even integer, prove that

$$\tan^4\frac{\pi}{2n}+\tan^4\frac{3\pi}{2n}+\ldots+\tan^4\frac{(n-1)\pi}{2n}=\frac{n^4-4n^2+3n}{6}.$$

8. Shew that the area of the triangle formed by the tangents to a rectangular hyperbola at the vertices of an inscribed equilateral triangle is half the area of the equilateral triangle.

9. If $S=0$ be the equation of any conic in trilinear coordinates, prove that the equation of the similar and similarly situated conic passing through three given points is

$$\begin{vmatrix} S & \alpha & \beta & \gamma \\ S' & \alpha' & \beta' & \gamma' \\ S'' & \alpha'' & \beta'' & \gamma'' \\ S''' & \alpha''' & \beta''' & \gamma''' \end{vmatrix} = 0,$$

where S' is the result of substituting the co-ordinates α', β', γ' in S, etc.

10. A system of co-planar forces has moments L, M, N about three given points A, B, C in the plane; shew that the system can be replaced by three forces $\frac{aL}{2\Delta}, \frac{bM}{2\Delta}, \frac{cN}{2\Delta}$ acting along BC, CA, AB, where a, b, c, Δ are the sides and area of the triangle ABC. Hence shew that if R is the resultant

$$R^2 = \frac{1}{4\Delta^2} \cdot \Sigma a^2 (L-M)(L-N).$$

11. A particle is projected from the lowest point O of a hollow sphere to strike the sphere at right angles at P. If α and θ be the angles which the direction of projection and OP make with the horizontal, prove that

$$2 \tan \alpha = \cot \theta + 3 \tan \theta.$$

12. If V is a function of x and y, and

$$x = u(1+v), \quad y = uv,$$

prove that

$$x \frac{\partial^2 V}{\partial x^2} - y \frac{\partial^2 V}{\partial y^2} = u \frac{\partial^2 V}{\partial u^2} - \frac{v}{u}(1+v) \frac{\partial^2 V}{\partial v^2}.$$

LXXIV.

1. If points A', B', C' be taken on the sides of a triangle, the circles on AA', BB', CC' as diameters have the orthocentre as their radical centre.

2. AA' is the major axis of an elliptic section of a given right circular cone whose vertex is V. Prove that the volume of the portion of the cone cut off by the plane of the section varies as

$$(VA\,.\,VA')^{\frac{3}{2}}.$$

3. Rationalise the equation

$$\{(y-z)^2+3x^2\}^{\frac{1}{2}}+\{(z-x)^2+3y^2\}^{\frac{1}{2}}+\{(x-y)^2+3z^2\}^{\frac{1}{2}}=0.$$

4. If the quantities a_1, $a_2 \dots a_n$ form an arithmetical progression, the common difference being r, prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \dots a_n \\ a_n & a_1 & a_2 \dots a_{n-1} \\ a_{n-1} & a_n & a_1 \dots a_{n-2} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_2 & a_3 \dots a_n & a_1 \end{vmatrix} = (-nr)^{n-1}\left(a_1+\frac{n-1}{2}r\right).$$

5. If every root of the equation $f'(x)=0$ be subtracted from every root of the equation $f(x)=0$, shew that the sum of the reciprocals of the differences is zero, provided the roots of $f(x)=0$ are all different.

6. Lines $AB'C'$, $BC'A'$, $CA'B'$ are drawn through the angular points A, B, C of a triangle making equal angles θ with AB, BC, CA respectively; and lines $AC''B''$, $CB''A''$, $BA''C''$ making equal angles θ with AC, CB, BA respectively. Shew that the triangles $A'B'C'$, $A''B''C''$ are equal in all respects, the area of each being

$$\Delta \sin^2\theta\,(\cot\theta-\cot A-\cot B-\cot C)^2.$$

7. Prove that

$$\frac{1}{e^2-1}=\frac{1}{\pi^2+1}+\frac{1}{4\pi^2+1}+\frac{1}{9\pi^2+1}+\dots.$$

8. All conics through the extremities of the principal axes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are cut orthogonally by the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2}.$$

9. The conic whose areal equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

meets the sides of the fundamental triangle in three pairs of points which are joined to the opposite vertices. Prove that the six lines so constructed touch the conic

$$\begin{vmatrix} bc, & -ch, & -bg, & x \\ -ch, & ca, & -af, & y \\ -bg, & -af, & ab, & z \\ x, & y, & z, & 0 \end{vmatrix}=0.$$

10. A smooth cone is placed vertex downwards in a circular horizontal hole. Prove that the position of equilibrium with the axis vertical is stable or unstable according as it is not, or is, the only possible position of equilibrium.

11. A gun is laid at an inclination θ with the intention that the projectile shall strike a point on a vertical cliff at an elevation α. It strikes at a small angle β too low. Shew that the gun should be raised through a small angle δ, where

$$\beta=\delta\,(1-\sec^2\theta\sin^2\overline{\theta-\alpha}).$$

12. Chords are drawn through the vertex of a parabola making angles α and β with the axis, and on the same side of the axis. Prove that the area of the included sector is

$$\tfrac{8}{3}\,a^2\,(\cot^3\beta\sim\cot^3\alpha),$$

where $4a$ is the latus-rectum.

LXXV.

1. From the vertices A, B, C of a triangle, perpendiculars AD, BE, CF are drawn to any straight line. Shew that the perpendiculars from D, E, F on BC, CA, AB respectively are concurrent.

2. Shew that the six intersections of three circles cannot lie on a conic unless the circles have their centres collinear.

3. The number of ways in which $3n$ things can be distributed in n groups of three each is

$$\frac{3\,.\,(3n-1)\,!}{6^n\,.\,(n-1)\,!}.$$

4. Sum to infinity the series

$$1+\frac{5x}{6}+\frac{13x^2}{15}+\frac{25x^3}{28}+\frac{41x^4}{45}+\ldots \qquad (x<1).$$

5. If three of the roots of the equation

$$x^4-px^3+qx^2-rx+s=0$$

are the tangents of the angles of a plane triangle, prove that the remaining root is

$$\frac{p-r}{1-q+s}.$$

6. The circle through the vertex A of a triangle which cuts off on AB and AC intercepts $b\cos^2\frac{A}{2}$ and $c\cos^2\frac{A}{2}$ respectively touches the inscribed circle and also the escribed circle opposite A.

7. A, B, C, D are consecutive angular points of a regular heptagon. Prove that one of the roots of the equation

$$x^3+1=2x^2+x$$

is the ratio $AD:AB$.

8. Three points on an ellipse (semi-axes a and b) are situated so that the circles of curvature at those points all cut the ellipse again at the same point. If their radii are ρ_1, ρ_2, ρ_3 shew that

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} + \rho_3^{\frac{2}{3}} = \frac{3\,(a^2+b^2)}{2a^{\frac{2}{3}}\,b^{\frac{2}{3}}}.$$

9. If a system of conics be drawn having four-point contact with

$$ax^2 + 2hxy + by^2 + 2fy = 0$$

at the origin, their director circles form a coaxal system with limiting points

$$(0, 0) \qquad \left(-\frac{hf}{a^2+h^2},\ \frac{hf}{a^2+h^2}\right).$$

10. Two equal smooth cylinders, radius a, are placed in contact on a smooth table, and a third smooth cylinder of radius b and weight W is placed symmetrically on them. A string is tied round the cylinders at such a tension T that there is no reaction between the lower cylinders. Prove that

$$T = \tfrac{1}{2}\,\frac{a}{(a-b) + \sqrt{b\,(2a+b)}}\,.\,W.$$

11. A rectangular billiard table has sides a and b, and its cushions have elasticity e. Shew that it is generally possible to choose a point in the side a such that a ball starting from it may describe a rhombus, and that, if θ be the angle which the direction of projection then makes with the side,

$$\tan\theta - e\cot\theta = \frac{(1+e)\,(b^2-a^2)}{2ab}.$$

12. Trace the curve

$$x^3 + 3xy^2 + 2a^2\,(y-x) = 0,$$

finding the equation of the line on which the points of inflexion lie.

LXXVI.

1. Find a pair of points on a given circle concyclic with each of two given pairs of points.

2. A rhombus is formed by joining the extremities of the axes of an ellipse, and a circle is inscribed in the rhombus. Prove that this circle touches every chord of the ellipse which subtends a right angle at the centre.

3. Prove that, if a_m and b_m are the coefficients of x^m in the expansions of $(1+x)^n$ and $(1-x)^{-k}$, n and k being positive integers, then

$$1 - \frac{a_1}{b_1} + \frac{a_2}{b_2} - \frac{a_3}{b_3} + \ldots = \frac{k-1}{k+n-1}.$$

4. Shew that the number of ways of dividing a number n into not more than three parts is

$$1 + I\left\{\frac{n(n+6)}{12}\right\},$$

where $I(x)$ denotes the greatest integer in x.

5. Prove that the equation

$$x^4 + qx^2 + rx + s = 0$$

cannot have equal roots unless $q^2 + 12s$ is positive. Prove also that if the equation has equal roots, the other two roots will be imaginary if $q^2 - 4s$ is negative.

6. Prove that in any triangle

$$\Sigma \frac{a}{\cos A + \sin \frac{A}{2} - \sin \frac{B}{2} - \sin \frac{C}{2}} = 0.$$

7. Shew that if n be an odd integer,

$$\cos^2 \frac{\pi}{n} + \cos^2 \frac{3\pi}{n} + \ldots + \cos^2 \frac{n-2}{n}\pi = \frac{n-2}{4}.$$

8. Tangents are drawn to a parabola from any point on a focal chord. Shew that the bisectors of the angles between them are parallel to the bisectors of the angles between the axis and the focal chord.

9. If the triangle of reference is equilateral, shew that $\beta+\gamma+3\alpha=0$ is a directrix of the conic $\beta^2+\gamma^2-3\alpha^2=0$.

10. A uniform triangular lamina ABC rests inside a smooth sphere. Prove that the pressures at the corners are equal, and if each is equal to X, and W is the weight of the lamina, R' the radius of the sphere, R that of the circumcircle of the lamina, then

$$\frac{W^2}{X^2}=9-\frac{8R^2}{R'^2}(1+\cos A\cos B\cos C).$$

11. Two particles, each of mass m, are moving with equal velocities at right angles to the line joining them. They are connected by a loose string to the middle point of which a second string is attached, at the other end of which is a particle of mass M which is at rest in the plane of the motion, and equidistant from the particles m. Prove that when the string tightens, the direction of motion of each of the particles m will be suddenly deflected through an angle

$$\tan^{-1}\frac{M\sin\alpha}{(2m-M)\cos\alpha+2m+M},$$

where α is the angle between the portions of the first string at the moment of tightening.

12. Prove that the radius of curvature of the envelope of the line $\alpha x+\beta y=1$, where α, β are functions of a single parameter t, is equal to

$$\frac{(\alpha^2+\beta^2)^{\frac{3}{2}}(\alpha''\beta'-\alpha'\beta'')}{(\alpha\beta'-\alpha'\beta)^3},$$

the accents denoting differentiations with regard to t.

LXXVII.

1. Three circles are touched internally by a circle whose centre is P and externally by a circle whose centre is Q. Shew that PQ passes through the radical centre of the three circles.

2. A variable rectangular hyperbola touches a given straight line at a given point, and has a given curvature there. Prove that the locus of its centre is a circle.

3. Prove that

$$\begin{vmatrix} 1, & 1, & 1 \dots & 1, & 1 \\ -1, & 2, & 0 \dots & 0, & 0 \\ 0, & -1, & 2 \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 \dots & -1, & 2 \end{vmatrix} = 2^n - 1,$$

the determinant having n rows and columns.

4. If n be a multiple of 6, the series

$$n - \frac{n(n-1)(n-2)}{3!} \cdot 3 + \frac{n(n-1)\dots(n-4)}{5!} \cdot 3^2 - \dots$$

and $$n - \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{3} + \frac{n(n-1)\dots(n-4)}{5!} \cdot \frac{1}{3^2} - \dots$$

will both vanish.

5. If the equation

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

whose roots are α, β, γ, be rendered reciprocal by the substitution $x = \lambda y + \mu$, then the three possible values of $a\mu + b$ are

$$\frac{b^2 - ac}{a\alpha + b}, \quad \frac{b^2 - ac}{a\beta + b}, \quad \frac{b^2 - ac}{a\gamma + b}.$$

6. In any triangle, prove that

$$(a + b - 2c)^2 \sec^2 \frac{C}{2} + (a - b)^2 \operatorname{cosec}^2 \frac{C}{2}$$

is equal to the two similar expressions, each being equal to $16OI^2$, where O is the circumcentre and I the incentre.

7. Prove that

$$\tan^4 \frac{\pi}{16} + \cot^4 \frac{\pi}{16} + \tan^4 \frac{2\pi}{16} + \cot^4 \frac{2\pi}{16} + \tan^4 \frac{3\pi}{16} + \cot^4 \frac{3\pi}{16} = 678.$$

8. A parabola of latus-rectum $4a$ rotates in its own plane about its vertex which is fixed. Shew that the locus of the point of intersection of tangents parallel to rectangular axes through the vertex is

$$x^{\frac{2}{3}} y^{\frac{2}{3}} (x^{\frac{2}{3}} + y^{\frac{2}{3}}) = a^2.$$

9. The equation to the circumcircle of the triangle formed by the lines

$$bx + cy = 1, \quad cx + ay = 1, \quad ax + by = 1,$$

is
$$\begin{vmatrix} (bx + cy - 1)^{-1}, & (cx + ay - 1)^{-1}, & (ax + by - 1)^{-1} \\ a(b - c), & b(c - a), & c(a - b) \\ bc + a^2, & ca + b^2, & ab + c^2 \end{vmatrix} = 0.$$

10. Four uniform rods whose weights are proportional to their lengths form a parallelogram $ABCD$, freely jointed at its angles. It is suspended from the point A and A, C are joined by a string of such a length that the figure is a rectangle. Find the tension of the string, and prove that the reactions at B and D are each equal to

$$\frac{wab}{\sqrt{2a^2 + 2b^2}},$$

where w is the weight of unit length of a rod.

11. Prove that if a particle be attached by two fine strings to two fixed points, the line joining which is inclined at an angle α to the vertical, and if the particle describe a circle of radius a with the strings taut, the square of the velocity at the highest point must be greater than $ga(\sin\alpha + \cos\alpha \cot\beta)$, where β is the angle the string to the higher of the two fixed points makes with the plane of the circle, supposed to lie between the two fixed points.

12. If r_1, r_2 denote the distances of any point on the lemniscate $r^2 = a^2 \cos 2\theta$ from its foci $\left(\pm \frac{a}{\sqrt{2}}, 0\right)$, and p_1, p_2 the perpendiculars drawn on the tangent at the point from these foci, prove that

$$\frac{4}{3\rho} = \frac{p_1}{r_1^2} + \frac{p_2}{r_2^2} = \sqrt{2}\left(\frac{1}{r_1} \sim \frac{1}{r_2}\right),$$

ρ being the radius of curvature at the point considered.

LXXVIII.

1. O is any point within a tetrahedron $ABCD$, and the lines AO, BO, CO, DO meet the opposite faces in A', B', C', D' respectively. Prove that

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} + \frac{OD'}{DD'} = 1.$$

2. What is the least value of the vertical angle of a cone in order that it may have a hyperbolic section of given eccentricity e?

3. On a flower stall there are five classes of flowers, viz. those which are sold singly, and those which are sold only in bunches of two, three, four and five respectively, all the flowers in any one bunch being of the same kind. In each class there are n different kinds of flowers. Shew that a selection of five flowers may be bought in

$$\frac{n(n+3)(n+6)(n^2+21n+8)}{120} \text{ ways.}$$

4. If n be a positive integer shew that

$$n^n = \sum_{r=1}^{r=n} \left[\frac{n!}{r!(n-r)!}\left\{r^n - r(r-1)^n + \frac{r(r-1)}{2!}(r-2)^n - \dots\right\}\right].$$

5. Solve the equations

$$x^2 + y^2 + z^2 = 0,$$
$$x^4 + y^4 + z^4 = 0,$$
$$x^5 + y^5 + z^5 = 2.$$

6. The area of the greatest equilateral triangle which can be drawn with its sides passing through three given points A, B, C is

$$2\Delta + \frac{a^2 + b^2 + c^2}{2\sqrt{3}},$$

where a, b, c are the sides, and Δ is the area, of the triangle ABC.

7. A regular polygon $ABC\ldots$ of n sides is inscribed in a circle. Prove that the sum of the products r together $(r < n)$ of PA^2, $PB^2\ldots$ is constant if P lies on a concentric circle.

8. The locus of the centres of conics touching the four straight lines

$$y = mx, \qquad y = \mu(x - c),$$
$$y = m'x, \qquad y = \mu'(x - c),$$

is the straight line

$$(2x - c)\left[mm'(\mu + \mu') - \mu\mu'(m + m')\right] = 2y(mm' - \mu\mu').$$

9. If e be the eccentricity of the conic whose trilinear equation is $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$, shew that

$$\frac{e^4}{1 - e^2} = \frac{l^2 + m^2 + n^2 + 2mn\cos 2A + 2nl\cos 2B + 2lm\cos 2C}{mn\sin^2 A + nl\sin^2 B + lm\sin^2 C}.$$

10. A rhombus $ABCD$ formed by four rods is suspended from A and held in shape by a weightless bar joining two points P, Q in AD, BC. Prove that the tension is

$$2W\frac{PQ \cdot AB}{AC(BQ - AP)},$$

where W is the weight of one rod.

11. A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Shew that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

12. In the curve $y^4 - axy^2 + x^4 = 0$, the radii of curvature at the points where the tangents are parallel to the axis of y are in the ratio $4 : 1$.

LXXIX.

1. A circle is described touching internally the circumcircle of a triangle ABC, and touching also each of the sides AB, AC. Prove that the length of the tangent to the circle from A is equal to $\frac{bc}{s}$, with the usual notation.

2. A circle has double contact with a hyperbola, and from P any point of the latter are drawn PM, parallel to an asymptote, to meet the chord of contact in M, and PT to touch the circle at T. Prove that $PM = PT$.

3. Shew that with n straight lines of lengths $1, 2, 3 \ldots n$ there can be formed $\frac{n(n-2)(2n-5)}{24}$ triangles if n is even, and

$$\frac{(n-3)(n-1)(2n-1)}{24}$$

if n is odd, excluding line-triangles.

4. If any integer n be divided by each of the $(n-1)$ integers less than n, the sum of the remainders together with the sum of all the factors (unity included) of all the numbers not greater than n is n^2.

5. If $ad^2 = b^2e$, shew that the roots of the equation

$$ax^4 - bx^3 + dx - e = 0$$

determine on a straight line four points forming a harmonic range.

6. If a, b, c, d be the lengths of the sides, taken in order, of a quadrilateral $ABCD$ in which the sum of the angles A and C is α, shew that the sides a and c are at right angles if

$$(a^2 + c^2)(b^2 + d^2) - 4abcd \cos \alpha - 2(a^2 - c^2) bd \sin \alpha = (b^2 - d^2)^2.$$

7. The roots of the equation

$$64x^3 - 192x^2 - 60x - 1 = 0$$

are $\qquad \cos^3 \frac{2\pi}{7} \sec \frac{6\pi}{7}, \quad \cos^3 \frac{4\pi}{7} \sec \frac{2\pi}{7}, \quad \cos^3 \frac{6\pi}{7} \sec \frac{4\pi}{7}.$

8. Shew that the straight line joining the middle points of perpendicular normal chords of the parabola $y^2 = 4ax$ touches the parabola

$$y^2 = 16a(5a - x).$$

9. A parabola is drawn having double contact with the conic $ax^2 + 2hxy + by^2 + 2y = 0$ along the line $lx + my = 0$. Prove that the curvature of the parabola at the origin is

$$-\frac{(am - hl)^2}{bl^2 + am^2 - 2hlm}.$$

10. A smooth hemisphere A is fixed with its base vertical, and an equal hollow smooth hemisphere B is fixed in contact with it with its base horizontal and vertex downwards, so that the point of contact of A and B is the vertex of A and is on the rim of B. A straight rod rests with one extremity on the convex surface of A and the other on the concave surface of B. Shew that its inclination λ to the horizontal is given by the equation

$$\cos\lambda = \frac{k^2 + 3 \pm \sqrt{2k^2 - 7}}{4k},$$

where k is the ratio of the length of the rod to the radius of either hemisphere.

11. Three equal balls A, B, C are moving with the same velocity v in directions inclined at angles 120° to one another and impinge, so that their centres form an equilateral triangle. If the coefficient of elasticity between C and either A or B is e, and between A and B is e', then A and B will separate with velocity

$$\frac{(2e' + e)v}{\sqrt{3}},$$

assuming that the compression ends at the same time for all the spheres.

12. Obtain the indefinite integral

$$\int \frac{dx}{\sin x(1 + \sin x + \cos x)}$$

in the form $$\frac{1}{2}\left(\tan\frac{x}{2} + \log\frac{\sin x}{1 + \sin x}\right).$$

LXXX.

1. Given the inscribed and circumscribed circles of a triangle in position, shew that the orthocentre lies on a fixed circle.

2. If two conics be inscribed in the same quadrilateral the two tangents at any of their points of intersection cut any diagonal of the quadrilateral harmonically.

3. Evaluate the determinant

$$\begin{vmatrix} 1, & a, & a^2, & a^3, & 0 \\ 0, & 1, & a, & a^2, & a^3 \\ a^3, & 0, & 1, & a, & a^2 \\ a^2, & a^3, & 0, & 1, & a \\ a, & a^2, & a^3, & 0, & 1 \end{vmatrix}.$$

4. Prove, by induction or otherwise, that

$$(2n)!\left[\frac{1}{2!}-\frac{1}{3!}+\ldots-\frac{1}{(2n-1)!}+\frac{1}{(2n)!}\right]$$

cannot be less than

$$[1 \,.\, 3 \,.\, 5 \ldots (2n-1)]^2.$$

5. If α, β, γ, δ be the roots of $x^4 - 4px^3 + 6qx^2 - 4rx + s = 0$, and if

$$\theta_1 = \tfrac{1}{2}(\alpha\beta + \gamma\delta), \qquad \theta_2 = \tfrac{1}{2}(\alpha\gamma + \beta\delta), \qquad \theta_3 = \tfrac{1}{2}(\alpha\delta + \beta\gamma),$$

prove that

$$(q-\theta_1)(q-\theta_2)(q-\theta_3) = 2\begin{vmatrix} 1 & p & q \\ p & q & r \\ q & r & s \end{vmatrix}.$$

6. Circles of radii $r, r', r'' \ldots r^{(n)}$ are drawn touching internally two sides of a triangle: each circle also touches its neighbours, and the first circle is the in-circle. Shew that

$$\sqrt[n]{r} = \sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho_2},$$

where ρ is written for $r^{(n)}$, and the suffixes 1, 2, 3 refer respectively to circles in the angles A, B, C.

7. If $x = \cot\theta$, prove that

$$\frac{x+h}{1+(x+h)^2} = \cos\theta\sin\theta - h\cos 2\theta\sin^2\theta + h^2\cos 3\theta\sin^3\theta - \dots.$$

8. Shew that the axes of the two parabolas which can be drawn through the four points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose eccentric angles are α, β, γ, δ, are inclined to one another at an angle

$$\tan^{-1}\left(\frac{2ab}{a^2-b^2}\operatorname{cosec}\frac{\alpha+\beta+\gamma+\delta}{2}\right).$$

9. Tangents are drawn at the feet of the normals from (f, g) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Shew that the parabola which touches these four tangents is

$$\sqrt{fx} + \sqrt{-gy} + \sqrt{a^2-b^2} = 0.$$

10. Two rods AB, AC are fixed in a vertical plane, each making an angle α with the horizontal. Two uniform rods PQ, $P'Q$ whose lengths are l and l' and weights W, W' are freely jointed at Q, and P, P' slide by smooth rings upon AB and AC. Prove by the method of virtual work that if these rods make angles θ, θ' with the horizontal, then

$$(W+W')\tan\theta = W'\cot\alpha,$$
$$(W+W')\tan\theta' = W\cot\alpha.$$

11. A smooth sphere of radius c falls so as to strike another equal sphere at rest on a smooth inelastic table. If after impact the first sphere begins to move horizontally, shew that when it impinges on the table it will have moved through a horizontal distance

$$\frac{2^{\frac{3}{4}}e^{\frac{1}{2}}c^{\frac{1}{2}}V}{g^{\frac{1}{2}}(2+e)^{\frac{1}{4}}},$$

where e is the coefficient of elasticity for the two spheres and V is the velocity of the falling sphere just before the first impact.

12. If the portion of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which lies in the first quadrant be rotated about either axis of co-ordinates, prove that the area of the surface generated is $\frac{6}{5}\pi a^2$.

LXXXI.

1. Four points lie on a circle. The pedal line of each of them with respect to the triangle formed by the other three is drawn. Prove that the four lines meet in a point.

2. If PP' be a chord perpendicular to the transverse axis of a rectangular hyperbola whose centre is C, prove that CP' is perpendicular to the tangent at P.

3. Prove that if

$$x + y + z = 0,$$

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0,$$

and

$$ayz + bzx + cxy = 0,$$

and x, y, z are not all zero, then

$$(b+c)(c+a)(a+b) = a^3 + b^3 + c^3 + 5abc.$$

4. If x, y, z, a, b, c be any real quantities such that $ax + by + cz = 0$, prove that the product

$$\frac{x^2 + y^2 + z^2 - yz - zx - xy}{(x+y+z)^2} \cdot \frac{a^2 + b^2 + c^2 - bc - ca - ab}{(a+b+c)^2}$$

is not less than $\frac{1}{4}$.

5. If the roots of the quadratic $ax^2 + bx + c = 0$ are real, then two of the roots of

$$\begin{vmatrix} x^3, & x^2, & x, & 1 \\ b, & a, & 0, & 0 \\ c, & b, & a, & 0 \\ 0, & c, & b, & a \end{vmatrix} = 0$$

are imaginary.

6. Prove that the polar circle of a triangle ABC intersects the circumcircle and the nine-point circle each at an angle

$$\cos^{-1}(-\cos A \cos B \cos C)^{\frac{1}{2}}.$$

7. Shew that if θ is small,

$$180\theta = 256 \text{ chd } \theta - 40 \text{ chd } 2\theta + \text{chd } 4\theta \text{ approximately,}$$

where chd θ is the length of a chord of a circle of unit radius which subtends an angle θ at the centre, and estimate the degree of accuracy of the approximation.

8. Shew that the equations of the common tangents of the circle $x^2 + y^2 = r^2$ and the circle whose diameter is the chord $x \cos \alpha + y \sin \alpha = p$ of the first circle are

$$(r - \sqrt{r^2 - p^2})(x \cos \alpha + y \sin \alpha)$$
$$\pm \sqrt{2p^2 - 2r^2 + 2r\sqrt{r^2 - p^2}}\,(y \cos \alpha - x \sin \alpha) = pr.$$

9. If a rectangular hyperbola be described about the triangle of reference, the normals at A and B will meet on the curve

$$\frac{(\gamma + \alpha \cos B) \cos A}{\alpha + \gamma \cos B} + \frac{(\gamma + \beta \cos A) \cos B}{\beta + \gamma \cos A} + \cos C = 0.$$

10. AD and BC are parallel straight lines, E and F their middle points. Prove that forces represented by AB, BC, CD, DA, AC, DB are equivalent to a single force represented in magnitude and direction by $2EF$, and find the distance from A of the point in which the line of action cuts AD.

11. Particles are projected from points on a straight line on a horizontal plane, so as to pass through a point at a height h and at a horizontal distance d from the line. Shew that if the horizontal component of the velocity of each particle is u, the particles will again reach the plane at points on a circle of radius

$$\frac{hu^2}{gd}.$$

12. Trace the curve $y^2(x + a) = x^2(x - a)$, and shew that the tangents at the points of inflexion are

$$5x \pm 3\sqrt{3}y - 4a = 0.$$

LXXXII.

1. Shew that the line joining the circumcentre and incentre of a triangle will pass through the orthocentre and centroid of the triangle formed by joining the points of contact of the inscribed circle.

2. If T is the pole of a chord PQ of a parabola which is normal at P, shew that the radius of curvature at P is equal to

$$2TP^2 \div PQ.$$

3. If the equations

$$\begin{aligned} ax_1x_2 + bx_1 + cx_2 + d &= 0,\\ ax_2x_3 + bx_2 + cx_3 + d &= 0,\\ ax_3x_4 + bx_3 + cx_4 + d &= 0,\\ ax_4x_1 + bx_4 + cx_1 + d &= 0, \end{aligned}$$

are satisfied by values of x_1, x_2, x_3, x_4 which are all different, then

$$b^2 + c^2 = 2ad.$$

4. Sum the series

$$\frac{x}{1\,(p+1)} + \frac{x^2}{2\,(p+2)} + \frac{x^3}{3\,(p+3)} + \dots \text{ to infinity}$$

when $x < 1$, and shew that when $x = 1$, its sum is

$$\frac{1}{p}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p}\right).$$

5. If x be a root of a cubic equation, shew that any rational function of x can be reduced to the form

$$\frac{ax + b}{cx + d},$$

where a, b, c, d are rational functions of the coefficients in the equation.

6. Prove that the equation

$$a + b \sin x = c \tan x$$

will be satisfied by two values of x between 0 and $-\frac{1}{2}\pi$, provided

$$b^{\frac{2}{3}} > a^{\frac{2}{3}} + c^{\frac{2}{3}}.$$

7. Prove that the real part of $i^{\log(1+i)}$ is

$$e^{-\frac{\pi^2}{8}} . \cos\left(\frac{\pi}{4}\log 2\right).$$

8. A circle is drawn to touch the parabola $y^2 = 4ax$ at a point P, and to pass through the other extremity Q of the focal chord PQ. It meets the parabola again in R. Shew that QR touches the curve

$$27ay^2 = 4(x + 2a)^3.$$

9. Shew that the trilinear equation to the ellipse through B and C which has one focus at the angular point A of the triangle of reference ABC, and the other focus in BC, is

$$\alpha^2 \sin^2 \frac{A}{2} + \beta\gamma + \gamma\alpha + \alpha\beta = 0.$$

10. Two equal uniform rods AB, AC, each of weight w, are freely jointed at A, and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane, and the middle points of the rods being joined by a light string. Shew that, if the string is stretched, its tension is

$$w(\tan \alpha - 2 \tan \beta),$$

where 2α is the angle between the rods, and β the angle either rod subtends at the centre.

11. A two-wheeled vehicle is being drawn along a level road with velocity v. The wheels (radius c) are connected by an axle (radius r) fixed to them: the weight of the vehicle, excluding the wheels and axle, is W, and its centre of mass is vertically above the middle point of the axle. Shew that, if the shafts are in a horizontal plane with the top of the wheels, the horse is working at a rate

$$\frac{Wvr \sin \lambda}{\sqrt{c^2 - r^2 \sin^2 \lambda}},$$

λ being the angle of friction between the axle and its bearings.

12. Shew that the radius of curvature of the evolute of $r^n = a^n \cos n\theta$, at the point corresponding to θ, is

$$\frac{n-1}{(n+1)^2} \cdot a \sin n\theta \,.\, (\cos n\theta)^{\frac{1-2n}{n}}.$$

LXXXIII.

1. If a tetrahedron has each of its pairs of opposite edges perpendicular to one another, the sphere which passes through the feet of the perpendiculars from the vertices on the opposite faces will also pass through the centres of gravity of the four faces.

2. A circle passes through two fixed points A and B, and cuts a fixed straight line in R, R'. Shew that the locus of the point of intersection of AR and BR' is a rectangular hyperbola.

3. If $a_1, a_2 \dots a_n$ be positive quantities, and A_r is the sum of their products taken r together, shew that

$$\frac{A_1}{c_1} \geqq \left(\frac{A_r}{c_r}\right)^{\frac{1}{r}},$$

where c_r is the number of products in A_r.

4. A and B play a chess match which is won by whoever first wins two games. The chances that A wins, draws or loses any game are a, b and c respectively. Prove that A's chance of winning the match is $a^2(a+3c)/(a+c)^3$.

5. If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are all real, shew that the difference between any two cannot exceed $3\sqrt{2(p^2-q)}$, and that the difference between the greatest and least is greater than

$$2\sqrt{\frac{p^2-q}{3}}.$$

6. Three circles of radii α, β, γ have their centres at the angular points A, B, C of the triangle ABC, and all three touch the same straight line. Shew that R, the radius of the circumcircle, must be given by one or other of four equations of which one is

$$-a^2b^2c^2 + 4R^2\{a^2(\alpha-\beta)(\alpha-\gamma) \\ + b^2(\beta-\gamma)(\beta-\alpha) + c^2(\gamma-\alpha)(\gamma-\beta)\} = 0.$$

7. Prove that

$$\cos^2 x + \cos^2\left(x + \frac{2\pi}{n}\right) + \dots + \cos^2\left[x + \frac{(2n-2)\pi}{n}\right] = \frac{n}{2}.$$

8. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse, the anharmonic ratio of the pencil subtended by them at any fifth point is

$$\frac{\sin \frac{1}{2}(\alpha-\beta) \sin \frac{1}{2}(\gamma-\delta)}{\sin \frac{1}{2}(\gamma-\beta) \sin \frac{1}{2}(\alpha-\delta)}.$$

9. Chords of the general conic subtend a right angle at a fixed point (x', y') on the curve. Shew that they pass through a point (X, Y) given by

$$(a+b)(X-x')+2(ax'+hy'+g)=0,$$
$$(a+b)(Y-y')+2(hx'+by'+f)=0.$$

10. A thin rigid hemispherical bowl, whose weight is w, is supported with its inner surface resting on a rough sphere of half its radius, the centres being in a vertical line. If a light scalepan be attached to a point in the rim and weights be placed in it till the bowl begins to slip, prove that the total weight in the pan when slipping takes place is

$$\frac{w}{2}\left(2 \sin \lambda - \sin \frac{\lambda}{2}\right) \Big/ \left(\cos \frac{\lambda}{2} - \sin \lambda\right),$$

where λ is the angle of friction.

11. A ring of mass m and a particle of mass m' are connected by a string of length l. The ring slides on a smooth rod. Initially the string is perpendicular to the rod and the particle is projected with velocity V parallel to the rod. Prove that when the string makes an angle θ with the rod, the square of its angular velocity is

$$mV^2/(m+m' \cos^2 \theta)\, l^2.$$

12. OAB is a quadrant of a circle, centre O and radius a, and C, D are the points of trisection of the arc AB. If the segment bounded by CD be revolved about OA, prove that the volume of the ring-shaped solid generated is

$$\tfrac{1}{12}(3\sqrt{3}-5)\pi a^3.$$

LXXXIV.

1. The faces of a pyramid are cut into n steps of equal height, whose faces are planes parallel and perpendicular to the base, whose projecting horizontal edges lie in the original faces of the pyramid, and whose projecting corners lie in the slant edges of the pyramid. Prove that if the altitude of the portion removed to form the top step is equal to the height of each step, the total volume cut away bears to the original volume the ratio

$$3n + 2 : 2(n+1)^2.$$

2. Construct an ellipse, given the centre and a self-conjugate triangle.

3. If x_1, y_1, z_1; x_2, y_2, z_2 are a pair of different solutions of

$$\frac{ax + hy + gz}{x} = \frac{hx + by + fz}{y} = \frac{gx + fy + cz}{z},$$

prove that

$$x_1x_2 + y_1y_2 + z_1z_2 = 0.$$

4. If x is positive, shew that

$$\log_e x > 1 - \frac{1}{x},$$

and hence shew that the limiting value of

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n},$$

when n is infinite, is $\log_e 2$.

5. Prove that the cubic

$$x^3 - 6x^2 + 11x + p = 0$$

has a pair of imaginary roots unless the absolute value of $p + 6$ is equal to or less than $\frac{2}{9}\sqrt{3}$.

6. If P be any point in the plane of a triangle whose orthocentre is K, then

$$PK^2 = \Sigma \cot B \cot C \,.\, PA^2 - 8R^2 \cos A \cos B \cos C.$$

7. Sum to infinity the series

$$\frac{\cos^3 a \cos 3a}{3\,!} + \frac{\cos^7 a \cos 7a}{7\,!} + \frac{\cos^{11} a \cos 11a}{11\,!} + \ldots.$$

8. Prove that the points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at which the chords of curvature pass through the point (X, Y) are its intersections with the circle

$$2(x^2+y^2) - \frac{a^2-b^2}{a^2} \cdot xX + \frac{a^2-b^2}{b^2} \cdot yY - a^2 - b^2 = 0.$$

9. Prove that if x_0, y_0, z_0 are the areal co-ordinates of the centre of the conic $lx^2 + my^2 + nz^2 = 0$, the squares of the semi-axes of the conic are the roots of the equation in λ

$$\begin{vmatrix} -2\lambda x_0^{-1}, & c^2, & b^2, & 1 \\ c^2, & -2\lambda y_0^{-1}, & a^2, & 1 \\ b^2, & a^2, & -2\lambda z_0^{-1}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

10. A rough hemisphere rests with its base on a horizontal plane. The rough lower end of a stick, sliding in a smooth vertical tube, rests on the hemisphere. Shew that the hemisphere begins to slide when the inclination a to the vertical of the radius to the lower end of the stick is given by

$$W_1 \tan(a - \lambda) = (W + W_1) \tan \lambda,$$

where λ is the angle of friction, and W, W_1 the weights of the hemisphere and stick respectively.

11. Two particles A, B of masses $2m$ and m respectively are tied to points of a string, one end of which, O, is fixed, and OA, AB are equal. The string being initially straight, B is projected at right angles to AB. Shew that the angle OAB is never less than a right angle, and also that when OAB is again a straight line, the velocity of B is half that of A.

12. Shew that from a given point P on a cubic, four tangents can be drawn to the cubic other than the tangent at P, and that their points of contact lie on a conic touching the cubic at P, and having at P a curvature half that of the cubic.

LXXXV.

1. Shew how to find the vanishing points of the two projective ranges traced by a variable tangent to a circle on two fixed tangents.

2. A number of conics having a common focus and equal latera-recta are described touching a given straight line. Shew that the locus of their remaining foci is an hyperbola.

3. Shew that, when n is an integer,

$$\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \dots$$
$$+ (2n-1)\left(\frac{2n+1}{2n-1}\right)^n \equiv n(2n+1).$$

4. Shew that

$$\frac{2^{37\times 73}-2}{37\times 73}$$

is a positive integer.

5. If α is an imaginary fifth root of unity, and if $x = \alpha - \alpha^4$, then

$$x^4 + 5x^2 + 5 = 0.$$

Express the remaining roots of this equation in terms of α.

6. If a circle, of radius ρ, touching internally the sides AB, AC of a triangle, have its centre at a point whose distances from the angular points A, B, C are x, y, z respectively, shew that

$$ax^2 + by^2 + cz^2 - abc = bc\,(b + c - a)\left(\frac{\rho}{r} - 1\right)^2,$$

where r is the radius of the inscribed circle of ABC.

7. If $\alpha = \dfrac{\pi}{14}$, prove that

$$\cos\alpha + \cos 3\alpha + \cos 9\alpha = \tfrac{1}{2}\sqrt{7},$$

and $$\cos 2\alpha + \cos 6\alpha + \cos 18\alpha = \tfrac{1}{2}.$$

8. Prove that two coplanar parabolas, whose axes are at right angles, have in general five finite common normals, which all touch a conic that touches the axes of the parabolas.

9. An ellipse passes through the fixed points $(\pm a, 0)$ and has double contact with the circle $x^2 + y^2 = c^2$. Shew that the locus of the foci is given by one or other of the equations

$$\frac{1}{a^2} = \frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{c^2 - r^2}, \quad \frac{1}{a^2} = \frac{\cos^2\theta}{c^2 + r^2} + \frac{\sin^2\theta}{c^2}.$$

10. A rough cylinder of radius a and weight W rests on the ground, and supports another rough cylinder of radius b and weight W' in contact with a vertical wall, the axes of the cylinders being horizontal and parallel to the wall. If 2α, 2β be the angles subtended by the cylinders at the foot of the wall, shew that for equilibrium to exist, the necessary coefficient of friction between the cylinders is not less than

$$(a \cot \alpha - b)/b\,(1 + \cot\beta);$$

between the upper cylinder and the wall is not less than unity, and between the lower cylinder and the ground is not less than

$$\frac{W'(a\cot\alpha - b)}{W(a\cot\alpha + b\cot\beta) + W'b\,(1 + \cot\beta)}.$$

11. A bead can slide on a rough straight wire which is rotating with angular velocity ω about a vertical axis intersecting it. Prove that in order that the bead may be in relative equilibrium, it must lie between two points of the wire whose distance apart is

$$\frac{g}{\omega^2}\,\frac{\tan(\alpha + \lambda) - \tan(\alpha - \lambda)}{\cos\alpha},$$

where α is the inclination of the wire to the horizontal.

12. Prove that

$$\text{(i)} \quad \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos^4\theta + \sin^4\theta} = \frac{\pi}{\sqrt{2}},$$

$$\text{(ii)} \quad \int_0^{\frac{\pi}{2}} \log\sin\theta\, d\theta = \frac{\pi}{2}\log\frac{1}{2}.$$

LXXXVI.

1. Prove that when the lengths of the sides of a quadrilateral are given, the angle between the diagonals is greatest when the quadrilateral is cyclic.

2. If the tangent at any point P of a hyperbola cut an asymptote in T, and if SP cut the same asymptote in Q, then $SQ = QT$, S being a focus.

3. Given that $x + y + z = 0$ and $ax^2 + by^2 + cz^2 = 0$, then must

$$x^n[(a+\lambda)^n - (a-\lambda)^n] + y^n[(b+\lambda)^n - (b-\lambda)^n] + z^n[(c+\lambda)^n - (c-\lambda)^n] = 0,$$

where λ^2 is a properly-chosen rational function of a, b, c.

4. If $n-1$ and $n+1$ are prime numbers greater than 5, prove that n is of the form $30m$ or $30m \pm 12$, and that

$$n^2(n^2+16) \equiv 0 \text{ (mod. 720)}.$$

5. Shew that if $(n-1)a_1{}^2 - 2na_2$ is negative, the roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

cannot be all real.

6. If a, b, c be three unequal quantities, the equations

$$\cot^{-1}(a+x) + \cot^{-1}(a+y) + \cot^{-1}(a+z) = \cot^{-1}a,$$

$$\cot^{-1}(b+x) + \cot^{-1}(b+y) + \cot^{-1}(b+z) = \cot^{-1}b,$$

$$\cot^{-1}(c+x) + \cot^{-1}(c+y) + \cot^{-1}(c+z) = \cot^{-1}c,$$

are not consistent unless

$$bc + ca + ab = 1.$$

7. If n be an odd integer, prove that

$$\frac{3^n - 1}{2n} = \left(4 + \cot^2\frac{\pi}{n}\right)\left(4 + \cot^2\frac{2\pi}{n}\right)\ldots\left\{4 + \cot^2\frac{(n-1)\pi}{2n}\right\}.$$

8. If $x \cos \alpha + y \sin \alpha = p$ touches

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then it cuts

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

at angles given by

$$(a^2 b^2 + p^2 \lambda) \tan^2 \theta - (a^2 - b^2) \lambda \sin 2\alpha \tan \theta - \lambda (\lambda + p^2) = 0.$$

Examine the cases $\lambda = -p^2$ and $\lambda = \pm ab$.

9. If ω be the angle between the axes, the equation of the tangent at the vertex of the parabola

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

is $$\left(\frac{a}{a \cos \omega + b} + \frac{b}{b \cos \omega + a}\right)\left(\frac{x \cos \omega + y}{a} + \frac{y \cos \omega + x}{b}\right) = 1.$$

10. A series of n equal uniform rods are jointed together, and one extremity of the series is fixed. Each rod, except the one with the fixed end, has a support placed under it, so that the rods rest in a horizontal position, and the pressures on the supports are all equal. Shew that the point of support of the rth rod from the fixed end divides its length in the ratio

$$r - 1 : 2n - r.$$

11. A particle is projected from a point A with the least velocity of projection with which it can pass through another point B, and the line AB makes an angle α with the vertical. Shew that the height above A of the highest point of the path is

$$AB \cos^4 \tfrac{1}{2}\alpha.$$

12. If $$e^{e^x} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

prove that

$$a_{n+1} = \frac{1}{n+1}\left(a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \ldots + \frac{a_{n-r}}{r!} + \ldots + \frac{a_0}{n!}\right),$$

and hence calculate the first six terms in the expansion.

LXXXVII.

1. Given the vertex of a triangle and the points in which the inscribed circle and the circle escribed to the base touch the base, construct the triangle.

2. If a tangent to a hyperbola is perpendicular to one of the asymptotes, then the part of the corresponding normal intercepted between the axes is bisected by the other asymptote.

3. If $x^5 + y^5 = 1$, prove that

$$\frac{5}{(x+y-1)^4} = 1 + \frac{y}{1-x} + \frac{x}{1-y} + \frac{x+y+x^2+y^2}{1-x-y+x^2+y^2+xy}.$$

4. If $\qquad (1 + c_1 x + c_2 x^2 + \ldots)^n = 1 + k_1 x + k_2 x^2 + \ldots,$

prove the relation

$$(r+1)\,k_{r+1} + (r-n)\,c_1 k_r + (r-1-2n)\,c_2 k_{r-1} \\ + (r-2-3n)\,c_3 k_{r-2} + \ldots = 0,$$

where terms containing negative suffixes are to be rejected, and $k_0 = 1$.

5. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$ax^4 - bx^3 + cx^2 - dx + e = 0,$$

prove that

$$a^4(\beta+\gamma+\delta-\alpha)(\gamma+\delta+\alpha-\beta)(\delta+\alpha+\beta-\gamma)(\alpha+\beta+\gamma-\delta) \\ = -b^4 + 4ab^2c - 8a^2bd + 16a^3e.$$

6. If x, y, z be the lengths cut off from the perpendiculars of a triangle ABC by the inscribed circle, shew that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} - \frac{1}{4r^2} = \frac{1}{8rR\cos A\cos B\cos C}.$$

7. Shew that the sum of the products of the fourth powers of the reciprocals of every pair of odd numbers is

$$\frac{\pi^8}{2^4 \,.\, 8\,!}.$$

8. If the parameters of three points on the parabola $y^2 = 4ax$ are the roots of the equation

$$m^3 + pm^2 + qm + r = 0,$$

prove that the equations to the circumcircles of the triangles formed by the three points and the tangents at the three points are respectively

$$x^2 + y^2 + (q - p^2 - 4)\, ax + \tfrac{1}{2}(r - pq)\, ay - a^2pr = 0,$$

and
$$x^2 + y^2 - (1 + q)\, ax + (p - r)\, ay + a^2q = 0.$$

9. The common self-conjugate triangle of the conics

$$ax^2 + by^2 = 1, \quad xy + gx + fy = 0$$

has for its vertices the points (x_1, y_1), (x_2, y_2), (x_3, y_3): prove that

$$\frac{a}{f}x_1x_2x_3 + \frac{b}{g}y_1y_2y_3 = 1.$$

10. If two particles of weights w_1, w_2 rest on a rough inclined plane of angle α, connected by a taut string which makes an angle θ with the line of greatest slope, and if the lower one (w_2) is on the point of motion, while the upper one is not, prove that

$$\mu_2 = \sin\theta \tan\alpha,$$

$$\mu_1 > \tan\alpha \left\{\sin^2\theta + \left(1 + \frac{w_2}{w_1}\right)^2 \cos^2\theta\right\}^{\frac{1}{2}},$$

where μ_1, μ_2 are the coefficients of friction of the particles.

11. Two equal balls are lying in contact on a smooth table, and a third equal ball, moving along their common tangent, strikes them simultaneously. Prove that $\frac{3}{5}(1 - e^2)$ of its kinetic energy is lost by the impact, e being the coefficient of restitution for each pair of balls.

12. If
$$x = \frac{\sec\theta}{\rho}, \quad y = \frac{\tan\theta}{\rho},$$

prove that

$$x^3\frac{\partial^3 V}{\partial x^3} + 3x^2y\frac{\partial^3 V}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 V}{\partial x \partial y^2} + y^3\frac{\partial^3 V}{\partial y^3} = -\rho^3\frac{\partial^3 V}{\partial \rho^3} - 6\rho^2\frac{\partial^2 V}{\partial \rho^2} - 6\rho\frac{\partial V}{\partial \rho}.$$

LXXXVIII.

1. The perpendiculars from the vertices of a tetrahedron $ABCD$ on the opposite faces intersect in a point O, and A', B', C', D' are the feet of those perpendiculars. Prove that the 'nine points' of the triangle ABC, the middle points of OA, OB, OC and the points A', B', C' lie on a sphere.

2. O, O', A, B, C, D are six points on a conic; OA, OB, OC, OD meet $O'B, O'C, O'D, O'A$ in P, Q, R, S respectively. Prove that if O, O', P, Q, R, S lie on a conic, then the pencil $O(ABCD)$ is harmonic.

3. Prove that

$$\begin{vmatrix} 1, & a, & a', & aa' \\ 1, & b, & b', & bb' \\ 1, & c, & c', & cc' \\ 1, & d, & d', & dd' \end{vmatrix} = \begin{vmatrix} 1, & bc+ad, & b'c'+a'd' \\ 1, & ac+bd, & a'c'+b'd' \\ 1, & ab+cd, & a'b'+c'd' \end{vmatrix}.$$

4. Shew that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\dots \text{ to } 3n \text{ terms}$$

is equal to

$$\frac{bc+1}{abc+a+c}+\frac{1}{(abc+a+b+c)}+\frac{1}{(abc+a+b+c)}+\dots$$

to n terms.

5. Prove that if $(ac-b^2)x^2+(ad-bc)x+(bd-c^2)$ be factorised in the form $(px+q)(p'x+q')$, then the transformation

$$x=\frac{q'y-q}{p-p'y}$$

reduces the cubic $ax^3+3bx^2+3cx+d=0$ to the form $Ay^3+D=0$.

6. The pedal line of any point on the circumcircle of the triangle ABC cuts the sides at distances x, y, z from the circumcentre. Prove that

$$\Sigma x^2 \sin 2A = (3R^2+OP^2)\sin A \sin B \sin C,$$

O being the circumcentre and P the orthocentre.

7. If $1+x$, $1-y$, x, y be in harmonical progression, and if $x+y$ be negative, prove that its value is

$$-\cos\frac{\pi}{9}.$$

8. If ABC is a triangle inscribed in an ellipse and the perpendiculars from A, B, C on the opposite sides meet the curve again in A', B', C' and AA', BB', CC' are the normals at A', B', C'; shew that if t_1, t_2, t_3 are the tangents of the eccentric angles at A', B', C',

$$3-4e^2+e^4+t_1t_2+t_2t_3+t_3t_1=0,$$

where e is the eccentricity of the ellipse.

9. The locus of the centres of equilateral triangles circumscribing the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the curve

$$9(x^2+y^2)^2-2(5a^2+3b^2)x^2-2(5b^2+3a^2)y^2+(a^2-b^2)^2=0.$$

10. An irregular polygon is circumscribed to a circle whose centre is O. If G be the centroid of the area of the polygon, and K that of its perimeter, prove that O, G, K are in one straight line, and that $OG=\frac{2}{3}OK$.

11. A heavy particle is attached to a fixed point by a fine string of length a: the particle is projected horizontally from the lowest point with velocity

$$\sqrt{ag\left(2+\frac{3\sqrt{3}}{2}\right)}.$$

Prove that the string will first become slack when inclined to the upward vertical at an angle 30°, will become tight again when horizontal, and slack again when inclined to the upward vertical at an angle

$$\cos^{-1}\left(\frac{3\sqrt{3}}{8}\right).$$

12. Find the length and area of the loop of the curve

$$3ay^2=x(a-x)^2,$$

and trace the curve.

LXXXIX.

1. Shew that in addition to the nine-point circle there are four circles which touch the escribed circles of a triangle, and that these are the inverses of the sides of the triangle and the nine-point circle with regard to the circle which cuts the escribed circles orthogonally.

2. The locus of the extremities of parallel diameters of a system of co-axial circles is a rectangular hyperbola.

3. Shew that the number of combinations, taken n together, of $3n$ letters of which n are a, n are b, and the rest are unlike, is

$$2^{n-1}(n+2).$$

4. On an average A wins four games to B's three. They play 5 games, all to a definite issue. Shew that the odds against A's winning three games in succession are 1569 : 832.

5. Shew that the product of the squares of the differences of the roots of the equation $x^n + ax + b = 0$ is

$$(-1)^{\frac{n(n+1)}{2}} \{-a^n (n-1)^{n-1} + (-1)^n b^{n-1} n^n\}.$$

6. The distances of the orthocentre of a triangle from the angular points are the roots of the cubic

$$(2R + x)(r + 2R - x)^2 = s^2 (2R - x).$$

7. Shew that

$$\tfrac{1}{2} \tan^{-1} (\sin \alpha \tan 2\beta) = \sin \alpha \tan \beta + \tfrac{1}{3} \sin 3\alpha \tan^3 \beta + \tfrac{1}{5} \sin 5\alpha \tan^5 \beta + \ldots.$$

8. If from the point (ξ, η) four normals are drawn to the rectangular hyperbola $4xy = a^2$, the corresponding tangents will all touch the parabola

$$(x\xi + y\eta)^2 - 2a^2 (x\eta + y\xi) + a^4 = 0.$$

9. A conic is reciprocated with regard to a point on it. Shew that it reciprocates into a parabola of latus-rectum $2c^2/\rho$, where ρ is the radius of curvature at P, and c is the constant of reciprocation.

10. Two small rings of weights W and W' can slide on a rough elliptical wire whose major axis is vertical. The rings are connected by a light string whose length is equal to the major axis and which passes over a smooth peg at the upper focus. Shew that if the rings are just on the point of motion their eccentric angles are given by

$$e^2 \sin^2 \phi = \mu^2 (1 - e^2) \left(\frac{W + W'}{W - W'}\right)^2,$$

where e is the eccentricity and μ the coefficient of friction.

11. A railway carriage of mass M moving with velocity v impinges on another of mass M' at rest. The force necessary to compress each buffer through its full extent l is equal to a weight of mass m. Assuming that the compression is proportional to the force, prove that the buffers will not be completely compressed if

$$v^2 < 2mgl \left(\frac{1}{M} + \frac{1}{M'}\right).$$

Also if v exceeds this limit, and the backing against which the buffers are driven is inelastic, the ratio of the final velocities is

$$Mv - \left\{2mM'gl\left(1 + \frac{M'}{M}\right)\right\}^{\frac{1}{2}} : Mv + \left\{2mMgl\left(1 + \frac{M}{M'}\right)\right\}^{\frac{1}{2}}.$$

12. If $y_2 = sy_1$, $y_3 = ty_1$, and all the letters denote functions of x, then

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1^3 \begin{vmatrix} s', & t' \\ s'', & t'' \end{vmatrix}.$$

XC.

1. Given three points A, B, C on a circle, find a fourth point D on the circle such that of the three rectangles $AD.BC$, $AB.CD$, $AC.BD$, the first may be a mean proportional between the other two.

2. A circle and a conic osculate at P and have diameters PQ, PR. Shew that the line QR, the tangent at P, and the other common tangent are concurrent.

3. Prove that if

$$v_r = \sum_{m=0}^{r-1} \frac{u_{r-m}}{m!},$$

then also

$$u_r = \sum_{m=0}^{r-1} (-1)^m \frac{v_{r-m}}{m!}.$$

4. If n is a positive integer, then

$$1 + n\frac{1}{2} + \frac{n(n+1)}{1.2}\frac{1}{2^2} + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}\frac{1}{2^r} + \dots$$
$$\dots\dots + \frac{n(n+1)\dots(2n-2)}{(n-1)!}\frac{1}{2^{n-1}} = 2^{n-1}.$$

5. Shew that the equation

$$2x^5 - 5px^2 + 3q = 0$$

has one or three real roots according as q^3 is $>$ or $\ngtr$ p^5, p and q being positive.

6. Prove that

$$\frac{1-\tan^2(x+a)\tan^2(y+a)}{1-\tan^2(x-a)\tan^2(y-a)} \cdot \frac{1-\tan^2 a\tan^2(x+y-a)}{1-\tan^2 a\tan^2(x+y+a)}$$
$$= \left\{\frac{1-\tan x\tan y\tan a\tan(x+y-a)}{1+\tan x\tan y\tan a\tan(x+y+a)}\right\}^2.$$

7. Prove that

$$\frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8} + \dots \text{ ad inf.} = \frac{\pi}{3\sqrt{3}}.$$

8. Two parabolas of latera-recta l and l' have the same vertex and their axes are inclined at an angle α. If at some other point they intersect at an angle $\alpha - \epsilon$, prove that

$$\left(\frac{l}{l'} + \frac{l'}{l}\right) \sec \alpha - 2 = \frac{9 \tan^3 \alpha \tan \epsilon \{3 \tan \alpha \tan^2 \epsilon - 4 (\tan \alpha - 2 \tan \epsilon)\}}{4 (\tan \alpha - 2 \tan \epsilon)^2 (2 \tan \alpha - \tan \epsilon)}.$$

9. Shew that the eccentricity e of the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

of which the side BC of the triangle of reference is a diameter, is given by

$$(2 - e^2)^2 mn \sin^2 A \sin B \sin C = (1 - e^2)(n \sin B + m \sin C)^2.$$

10. Three uniform rods OA, OB, OC, each of length a, are freely jointed at a fixed point O, and have their upper extremities joined by uniform elastic strings each of natural length a. If the rods are equally inclined to the vertical and a sphere of radius b is placed between them, and if the weight of the sphere and the modulus of the strings are each equal to the weight of a rod, then if the strings are not in contact with the sphere, the inclination of the rods to the vertical is given by

$$2b \operatorname{cosec}^3 \theta = 3a (6 - \sec \theta - 2\sqrt{3} \operatorname{cosec} \theta).$$

11. A perfectly elastic particle is to be projected from a point A on the floor of a rectangular room with smooth vertical walls so as to reach another point B on the floor, after striking two adjacent walls in succession, but neither the floor nor ceiling. Shew that if the distances of A and B from the two walls are a, b and b, a respectively and the height of the room c, then the minimum height to which the velocity is due must be $\frac{1}{2}(a + b)\sqrt{2}$, if this is less than $2c$, otherwise it must be $c + \frac{1}{8}(a + b)^2/c$.

12. If the portion of the curve

$$a^2 y = x^3$$

between $x = 0$ and $x = a$ be revolved about the axis of x, prove that the area of the surface generated is

$$\frac{\pi a^2}{27}(10\sqrt{10} - 1).$$

XCI.

1. Three circles cut one another orthogonally at the three pairs of points AA', BB', CC'. Prove that the circles ABC and $AB'C'$ touch at A.

2. A is a given point in the plane of a given circle and ABC a given angle. If B moves round the circumference of the circle, prove that for different values of the angle ABC, the envelopes of BC are similar conics, and that all their directrices pass through one or other of two fixed points.

3. If of $3n$ letters, n are alike of each of three kinds, and if $r > n$ and $< 2n+1$, the number of combinations r together is

$$\tfrac{1}{2}(n+1)(n+2)+(r-n)(2n-r).$$

4. Prove that $(a^2-b^2)\sqrt{ab}$ is divisible by 240, where a and b are integers such that ab is a perfect square, and $a-b$ is even.

5. The sum of the squares of two roots of the equation

$$x^3+p_1x^2+p_2x+p_3=0$$

is equal to their product. Prove that

$$8p_3^2-p_1^2p_2^2-10p_1p_2p_3+3p_1^3p_3+3p_2^3=0.$$

Hence solve the equation

$$x^3-6ax^2+81a^3=0.$$

6. If α, β, γ, δ be the distances of the nine-point centre from those of the inscribed and escribed circles of a triangle, prove that

$$\Sigma\frac{1}{\beta+\gamma+\delta-11\alpha}=0,$$

and

$$\Sigma\alpha^2=R^2(13-8\cos A\cos B\cos C).$$

7. Prove that the roots of the equation

$$x^4 - 28x^3 + 70x^2 - 28x + 1 = 0$$

are $\tan^2 \frac{r\pi}{16}$, where $r = 1, 3, 5, 7$.

8. The conic circumscribing the two triangles formed by the tangents to the parabola $y^2 = 4ax$ at the points whose ordinates are y_1, y_2, y_3 and y_1', y_2', y_3' respectively passes through the vertex of the parabola. Shew that

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} = \frac{1}{y_1'} + \frac{1}{y_2'} + \frac{1}{y_3'}.$$

9. Shew that the radius of the circle

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = k\alpha\,(a\alpha + b\beta + c\gamma)$$

is $R\sqrt{1 - 2k\cos A + k^2}$, where R is the radius of the circumscribing circle of the triangle of reference.

10. A regular heptagon of heavy rods $ABCDEFG$, freely jointed, has smooth rings attached to the joints A and D, and the rings rest on a fixed horizontal rod. Prove that in equilibrium the tangent of the inclination of the rods EF, FG to the horizon is given by the equation

$$(1 + 9x^2)^{-\frac{1}{2}} - (1 + 4x^2)^{-\frac{1}{2}} + (1 + x^2)^{-\frac{1}{2}} = \tfrac{1}{2}.$$

11. A shot is fired horizontally in a direction due east from the top of a tower c feet high, and another shot is fired at the same instant so as to strike the former from a point on the ground a feet to the east and b feet to the south of the tower. If the initial velocities are the same, shew that the direction in which the latter shot is fired is inclined at angles $\cos^{-1} \frac{2ac}{a^2 + b^2 + c^2}$ to the vertical and $\cos^{-1} \frac{b^2 + c^2 - a^2}{a^2 + b^2 + c^2}$ to a line due east.

12. Shew that there are just two real lines which are both tangent and normal to the curve $y^2 = x^3$ and that the abscissae of the points where either of them touches the curve and cuts it at right angles are $\frac{8}{9}$ and $\frac{2}{9}$ respectively.

XCII.

1. A fixed point O external to a given circle is joined to the extremities A, B of any diameter and OA, OB meet the circle again in P, Q. Shew that the tangents at P and Q intersect on a fixed line parallel to the polar of O.

2. If an ellipse has a given focus and touches two fixed straight lines, the director circle passes through two fixed points.

3. Shew that

$$\begin{vmatrix} (b+c)^3, & -c^3\ , & -b^3 \\ -c^3\ , & (c+a)^3, & -a^3 \\ -b^3\ , & -a^3\ , & (a+b)^3 \end{vmatrix}$$

$$= 3\,(bc+ca+ab)^3\left[bc\,(b+c)+ca\,(c+a)+ab\,(a+b)\right].$$

4. Prove that 437 is a factor of $16^{99}-1$, and also a factor of $1+18!$.

5. Calculate the value of the symmetric function

$$\Sigma\,(\alpha-\beta)^2\,(\alpha-\gamma)^2\,(\alpha-\delta)^2\,(\alpha-\epsilon)^2$$

for the quintic

$$x^5-5ax-1=0.$$

6. Prove that

$$\cos\frac{2\pi}{n}+2\cos\frac{4\pi}{n}+3\cos\frac{6\pi}{n}+\dots+(n-1)\cos\frac{2\,(n-1)\,\pi}{n}=-\tfrac{1}{2}n.$$

7. Shew that $1+\dfrac{1}{2}+\dfrac{1}{3}+\dots+\dfrac{1}{n}-\log n$ has, when n is infinite, a finite limit lying between

$$\frac{\pi^2}{12}\text{ and }\frac{\pi^2}{12}-\frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\dots\text{ ad inf.}\right).$$

8. A chord PQ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P, a point whose eccentric angle is α. A parabola is drawn touching the ellipse at P and Q. Shew that the equation of its axis is

$$(b^6\cos^2\alpha+a^6\sin^2\alpha)\,(b^3\cos\alpha\,.\,x+a^3\sin\alpha\,.\,y)$$
$$=a^3b^3\,(a^2-b^2)^2\sin^2\alpha\cos^2\alpha.$$

9. The conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is reciprocated with respect to a point. If the reciprocal be always similar to the original conic, prove that the point lies on the curve

$$a^2b^2\,(x^2+y^2)^2=(a^4-b^4)\,(b^2x^2-a^2y^2).$$

10. A rod of weight W has one end on a rough horizontal floor and rests against the rough horizontal edge of a table, the other end being acted on by a horizontal force P in a direction parallel to the edge of the table. If the rod is on the point of slipping along the floor, but not along the table, prove that

$$P=\frac{\mu\,(1-n)}{(2n-1)\cos\beta\,(1+\mu\tan\alpha\sin\beta)}\,W,$$

where μ is the coefficient of friction, α the inclination to the horizon of the plane containing the rod and the edge, β the inclination of the vertical plane containing the rod to that perpendicular to the edge, and $n\,(<1)$ is the ratio of half the length of the rod to the distance between the points of contact.

11. An ellipse has its major axis $(2a)$ vertical and a particle starting from rest at the vertex falls down the outside of the curve (supposed smooth), and leaves it at a point whose eccentric angle is ϕ. Shew that it will proceed to describe a parabola of latus-rectum

$$2a\,(1-\cos\phi)^2\,(2+\cos\phi).$$

12. Evaluate the integrals

$$\int e^x\,\frac{1+\sin x}{1+\cos x}\,dx \quad\text{and}\quad \int\frac{\sin x}{\sin(x-\alpha)\sin(x-\beta)}\,dx.$$

XCIII.

1. Prove that the operations of inversion with respect to two circles are commutative only when the circles cut orthogonally.

2. From any point P on a given diameter of a hyperbola, two straight lines are drawn parallel to the asymptotes meeting the hyperbola in Q and Q'. Prove that PQ, PQ' are in a constant ratio.

3. Prove that the result of eliminating x between

$$ax^2 + bx + c = 0$$

and

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

is

$$\{a^7 + b^7 + c^7 - 7abc(b^2 - ac)^2\}(a + b + c)^{-1} = 0.$$

4. Prove that the even convergents of the continued fraction

$$\frac{2}{1+}\,\frac{3 \,.\, 1^3}{1+}\,\frac{4 \,.\, 2^3}{1+}\,\frac{5 \,.\, 3^3}{1+}\cdots$$

tend to $\log 2$ as a limit.

5. Shew that the equation which has for roots the values of

$$\frac{\beta\gamma(\alpha+\delta) - (\beta+\gamma)\alpha\delta}{\beta+\gamma-\alpha-\delta}$$

where $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$ is

$$\lambda^3(p^3 - 4pq + 8r) - \lambda^2(rp^2 - 4qr + 8ps) \\ - \lambda(pr^2 - 4pqs + 8rs) + r^3 - 4qrs + 8ps^2 = 0.$$

6. If

$$\begin{vmatrix} \cos^6\alpha, & \sin^6\alpha, & 1 \\ \cos^6\beta, & \sin^6\beta, & 1 \\ \cos^6\gamma, & \sin^6\gamma, & 1 \end{vmatrix} = 0,$$

where α, β, γ are three real angles, prove that at least two of the quantities $\cos 2\alpha$, $\cos 2\beta$, $\cos 2\gamma$ are equal.

7. Prove that the sum to infinity of the series

$$\tan^{-1}p^2 + \tan^{-1}\frac{p^2}{4} + \tan^{-1}\frac{p^2}{9} + \dots$$

is
$$n\pi + \tan^{-1}\left(\frac{\tan\theta - \tanh\theta}{\tan\theta + \tanh\theta}\right),$$
where $\theta = p\pi/\sqrt{2}$, and n is some integer.

8. From any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, three normals (other than the normal at the point) are drawn, and their feet are P, Q, R. Shew that the sides of the triangle PQR touch the ellipse
$$\frac{x^2}{a^6} + \frac{y^2}{b^6} = \frac{1}{(a^2 - b^2)^2}.$$

9. If the triangle of reference be equilateral, the locus of the foci of conics touching the four lines $l\alpha \pm m\beta \pm n\gamma = 0$ is
$$2(l^2 + m^2 + n^2)\,\alpha\beta\gamma = l^2\alpha^2(\alpha - \beta - \gamma) + m^2\beta^2(\beta - \gamma - \alpha) + n^2\gamma^2(\gamma - \alpha - \beta).$$

10. A gipsy's tripod consists of three equal rods, each of length c and weight W, freely jointed at their upper ends. The lower ends rest on a smooth horizontal plane and form an equilateral triangle. The rods are prevented from separating by three horizontal wires each of length l, attached at equal distances from the feet, and forming a horizontal triangle. The apex of the tripod is at a height h from the plane, and supports a kettle of weight $3W$. Prove that the tension of each wire is
$$\frac{3(c^2 - h^2)}{2hl}\,W.$$

11. A shot is projected from a point on a hillside with a velocity due to a height h. Prove that the area commanded is $4\pi h^2 \sec^3\alpha$, the hill being a plane inclined at an angle α to the horizon.

12. The tangent at a point of the cardioid
$$r = a(1 + \cos\theta)$$
whose vectorial angle is 2α meets the curve again at a point whose vectorial angle is 2β. Shew that
$$\cos(2\beta - \alpha) + 2\cos\alpha = 0.$$

XCIV.

1. Through two given points P and Q construct a pair of parallel lines, which shall intersect two fixed lines in points A and B, such that the line AB passes through a third fixed point C. How many solutions are there?

2. A chord PQ is normal to a rectangular hyperbola at P, and another chord RS is drawn parallel to PQ. Shew that PR and QS intersect on the diameter perpendicular to CP.

3. Prove that the integral part of $(\sqrt{5}+2)^n$ is of the form $4r$ or $4r+1$ according as n is odd or even.

4. If n be any integer, and $\alpha, \beta, \gamma, \dots$ the prime numbers in the series $2, 3, 4, \dots n$, prove that

$$n! < \alpha^{\frac{n}{\alpha-1}} . \beta^{\frac{n}{\beta-1}} . \gamma^{\frac{n}{\gamma-1}} \dots\dots$$

5. Prove that, for the equation $x^5 - 5p^3x^2 + 5p^2x - q = 0$, the value of the function $\Sigma\, a_1{}^4 a_2{}^3 a_3{}^2 a_4$ is $-(5q^2 + 500p^5)$.

6. Shew that the lengths of the straight lines joining the middle points of opposite sides of a quadrilateral are roots of the equation

$$x^4 - (\sigma^2 + \tau^2)\, x^2 + \sigma^2\tau^2 + \Delta^2 = 0,$$

where 2σ is the sum of the diagonals, 2τ their difference, and Δ the area of the quadrilateral.

7. Prove that, if θ lies between $\pm\frac{1}{3}\pi$, then

$$\sin\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{1}{4}\sin 4\theta - \tfrac{1}{5}\sin 5\theta + \tfrac{1}{7}\sin 7\theta + \dots$$
$$= \frac{1}{\sqrt{3}} \log \frac{1+\sqrt{3}\tan\frac{1}{2}\theta}{1-\sqrt{3}\tan\frac{1}{2}\theta}$$

where in the series multiples of 3θ are omitted.

8. If the tangents drawn from a point to the circle

$$x^2 + y^2 + 2ax + c^2 = 0$$

are conjugate lines with respect to the circle

$$x^2 + y^2 + 2bx + c^2 = 0,$$

shew that the locus of the point is

$$(a^2 + b^2 - 2c^2)(x^2 + c^2) + 2(ab - c^2)\{y^2 + (a + b)x\} = 0.$$

9. Shew that the equation of the conic which touches the sides of the triangle of reference, and is confocal with the conic $\beta\gamma + \gamma\alpha + \alpha\beta = 0$, is

$$\sqrt{\alpha}\sin\tfrac{1}{2}A + \sqrt{\beta}\sin\tfrac{1}{2}B + \sqrt{\gamma}\sin\tfrac{1}{2}C = 0.$$

10. A solid is formed from a hemisphere, vertex A and centre O, by dividing it along the plane bisecting OA at right angles. The portion containing A rests in equilibrium with its curved surface in contact with a perfectly rough inclined plane of angle α. Shew that the plane face of the solid will make with the horizontal an angle

$$\sin^{-1}\left(\frac{40\sin\alpha}{27}\right).$$

11. A smooth wedge of mass M and angle α is free to slide on a horizontal plane in the direction of the projection of the lines of greatest slope. The wedge is held at rest while a particle of mass m is projected with a velocity V up its face in a direction making an angle β with the horizontal lines on the wedge, and is then immediately released. Shew that the path of the particle on the face of the wedge will be a parabola of latus-rectum

$$\frac{2V^2\cos^2\beta\,(M + m\sin^2\alpha)}{g\sin\alpha\,(M + m)}.$$

12. The circle of curvature at a point P of a parabola meets the curve again in Q, and its centre is C. Prove that the area of the triangle CPQ is a maximum when CP makes with the axis an angle

$$\tan^{-1}\left(\frac{1}{\sqrt[4]{5}}\right).$$

XCV.

1. Any point O is taken in the plane of a triangle ABC, and through D, E, F, which are the middle points of BC, CA, AB respectively, lines are drawn parallel to OA, OB, OC respectively. Prove that these lines meet in a point.

2. Two circles are drawn each having double contact with a parabola. Prove that their centres of similitude are equidistant from the focus.

3. Prove that the ratio of n^2 to the arithmetic mean of the products of all distinct pairs of positive integers whose sum is n tends to the limit 6 as n increases.

4. Shew that the total number of signals which can be made with n different flags on r different masts is one less than the coefficient of x^n in the expansion of

$$n!\, e^x (1-x)^{-r}.$$

5. Shew that the sum of the fourth powers of the differences of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$$

is

$$(n-1)\, p_1^4 - 4np_1^2 p_2 + 4\,(n-3)\, p_1 p_3 + 2\,(n+6)\, p_2^2 - 4np_4.$$

6. Prove that if the equation

$$\cos 3\theta + a \cos 2\theta + b \cos \theta + c = 0$$

is satisfied by each of the angles of a real triangle, then

$$a^2 + 2a - 2b - 2c + 2 = 0.$$

What other conditions are necessary?

7. If $$\tan(\phi + \alpha) = \frac{3 \tan \alpha + \tan^3 \alpha}{1 + 3 \tan^2 \alpha},$$

shew that one value of ϕ is the series

$$\frac{1}{1\,.\,3} \sin 4\alpha + \frac{1}{2\,.\,3^2} \sin 8\alpha + \frac{1}{3\,.\,3^3} \sin 12\alpha + \ldots.$$

8. An ellipse of constant eccentricity e osculates the parabola $y^2 = 4ax$. Prove that the locus of its centre is given by the equation

$$(1-e^2)(y^2+4ax+8a^2)^2 = 8a^2(2-e^2)^2(4ax-y^2).$$

9. The conic similar and similarly situated to the general conic and circumscribing the triangle of reference is

$$a(2u'bc - vc^2 - wb^2)\beta\gamma + b(2v'ca - wa^2 - uc^2)\gamma\alpha + c(2w'ab - ub^2 - va^2)\alpha\beta = 0.$$

Deduce the conditions that the general conic may represent a circle.

10. A rod of length $2ae$ rests symmetrically within a rough elliptical hoop with its major axis horizontal. If the hoop is turned slowly round in a vertical plane, prove that the rod will begin to slip when it has turned through an angle θ given by

$$\tan\theta = \frac{\mu(1+e^2)}{1-\mu^2e^2},$$

where a, e, μ have their usual meanings.

11. A particle is projected from a point on the outside of a circular hoop standing in a vertical plane, the direction of projection being upwards along the tangent to the circle. Shew that if the particle is to clear the hoop, the velocity of projection must exceed $\sqrt{ag\sec\alpha}$, where a is the radius and α the angular distance of the point of projection from the highest point.

12. Trace the curve

$$a(x^2+y^2) = xy(x+y),$$

and shew that the circle

$$x^2+y^2+2a(1+\sqrt{2})(x+y-2a) = 2a^2$$

has treble contact with it.

XCVI.

1. Given three circles in a plane, shew how to invert them simultaneously into great circles of a sphere.

2. A parabola and an ellipse have a common latus-rectum. Prove that it bisects the angle between the distances from the focus to the points of contact of common tangents.

3. Shew that the sum of the first n terms of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ lies between the limits $n\{(1+n)^{\frac{1}{n}} - 1\}$ and

$$n\left\{1 - (1+n)^{-\frac{1}{n}} + \frac{1}{1+n}\right\}.$$

4. Two queens are placed at random on a chess-board; prove that the chance that they cannot take one another is $\frac{23}{36}$.

5. Shew that, if α, β, γ, δ be the roots of the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

then the equation

$$\sqrt[3]{\beta\gamma + \alpha\delta - x} + \sqrt[3]{\gamma\alpha + \beta\delta - x} + \sqrt[3]{\alpha\beta + \gamma\delta - x} = 0$$

has only one root, viz.,

$$x = \frac{2}{a}\left(\frac{2J}{I} + c\right),$$

where $\quad I = ae - 4bd + 3c^2, \quad J = ace + 2bcd - ad^2 - eb^2 - c^3.$

6. A triangle ABC is placed so that the height of each vertex above a horizontal plane is equal to the perpendicular on the opposite side. Prove that the inclination of the plane of the triangle to the horizontal is θ, where

$$\cos 2\theta = 4(\cos A + \cos B + \cos C) - 5.$$

7. Prove that

$$s! = 2^{\frac{1}{2}s(s-1)} \cdot \prod_{n=2}^{n=s} \prod_{m=1}^{m=n-1} \sin \frac{m}{n}\pi.$$

8. Shew that the normals drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from any point on the curve $(c^4 - a^2x^2 - b^2y^2)^3 = 54a^2b^2c^4x^2y^2$, $(c^2 \equiv a^2 - b^2)$, form a harmonic pencil.

9. Shew that the equation of the two directrices perpendicular to an axis of length $2a$ of the conic $S=0$, is

$$\Delta S + a^2 CK = 0,$$

where $K \equiv C(x^2+y^2) - 2Gx - 2Fy + A + B$, and A, B, C.. have their usual meanings.

10. Two uniform straight rods PQ, $P'Q$ in all respects alike are smoothly jointed at Q, and at P, P' carry small rings which slide on a smooth fixed parabolic wire, whose axis is vertical and vertex upwards. Prove that in the symmetrical position of equilibrium the angle either rod makes with the horizontal is

$$\sin^{-1}\left\{\frac{aW}{l(W+w)}\right\},$$

where W is the weight of either rod, w of either ring, l the length of either rod and $4a$ the latus-rectum of the parabola.

11. A heavy particle, perfectly elastic, falls down a chord from the highest point of a vertical circle, and after reflection at the curve, passes through the lowest point. Prove that the inclination of the chord to the vertical is

$$\tfrac{1}{2}\cos^{-1}\left(\frac{\sqrt{3}-1}{4}\right).$$

12. Shew that the integral

$$\int \frac{dx}{(x-a)^2\sqrt{(x-b)(x-c)}}$$

can be rationalised by the substitution

$$y^2 = \frac{x-c}{x-b},$$

and hence evaluate the integral in the case $a<b<c$.

XCVII.

1. Two spheres of radii a and b have their centres distant c $(< a + b)$ apart. Prove that, if one sphere does not lie entirely within the other, the volume common to the spheres is

$$\frac{\pi}{12c}(a+b-c)^2\left[c^2+2(a+b)c-3(a-b)^2\right].$$

2. Given three points on a hyperbola and the directions of both asymptotes, construct the asymptotes.

3. The coefficient of x^r in the expansion of $(1+x)^n$ is c_r. Shew that

$$1^3 . c_1{}^2 + 2^3 . c_2{}^2 + 3^3 . c_3{}^2 + \ldots = \frac{n^2(n^2-1).(2n-3)!}{\{(n-1)!\}^2}.$$

4. Shew that

$$\begin{vmatrix} x^{n-1}, & x^{n-2} & \ldots & x, & 1 \\ 1, & x^{n-1} & \ldots & x^2, & x \\ x, & 1 & \ldots & x^3, & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{n-2}, & x^{n-3} & \ldots & 1, & x^{n-1} \end{vmatrix} = (x^n-1)^{n-1}.$$

If all the elements below the leading diagonal have negative signs, prove that the value of the determinant is $(x^n+1)^{n-1}$.

5. If s_n be the sum of the n^{th} powers of the roots of the equation $$ax^4+4bx^3+6cx^2+4dx+e=0,$$
shew that
$$\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$
is a numerical multiple of

$$\frac{1}{a^4}\{2(ac-b^2)(ae-4bd+3c^2)-3a(ace+2bcd-ad^2-eb^2-c^3)\}.$$

6. If a, b, c, d be the sides of a quadrilateral taken in order, and x and y its diagonals, shew that the line joining the middle points of a and c makes with the line joining the middle points of the diagonals an angle

$$\cos^{-1}\frac{b^2-d^2}{\{(b^2+d^2)^2-(x^2+y^2-a^2-c^2)^2\}^{\frac{1}{2}}}.$$

7. Through two fixed points A and B at a distance c apart, are drawn n different circles, such that AB subtends at points on them angles which form an arithmetic progression of difference π/n. If with A as centre, and radius a, a circle be drawn cutting these circles, the product of the n common chords is

$$2a^n \operatorname{sech}^n \alpha \,(\cosh^2 n\alpha - \cos^2 n\theta)^{\frac{1}{2}},$$

where θ is the complement of any one of the angles, and $\cosh \alpha = c/a$.

8. Shew that the locus of a point P which moves so that

$$\lambda \,.\, PA^2 + \mu \,.\, PB^2 + \nu \,.\, PC^2 = 0,$$

where λ, μ, ν are constants, is a circle which cuts the circumcircle of the triangle ABC orthogonally.

9. If the equation $lx^2 + my^2 + nz^2 = 0$ in areal coordinates represents an ellipse, prove that the ratio of the area of the triangle of reference to the area of the ellipse is

$$(mn + nl + lm)^{\frac{3}{2}} : -2\pi lmn.$$

10. Five equal uniform heavy rods AB, BC, CD, DE, EF, freely jointed together, have the ends A and F at fixed points in the same horizontal line and the distance AF is twice the length of a rod. They are in equilibrium with CD horizontal and with AB, EF and BC, DE respectively, equally inclined to the horizontal. Prove that if θ is the inclination of AB or EF to the horizontal, $\cos\theta$ is a root of the equation

$$(1 - 2x)\sqrt{3x^2 + 1} = 4x.$$

11. A pulley of negligible mass is hung from a fixed point by an elastic spring of modulus λ and natural length a, and an inextensible string passing over the pulley carries at its ends particles of masses m and m'. Prove that the time of a small vertical oscillation of the pulley is

$$4\pi \sqrt{\frac{mm'}{m + m'} \cdot \frac{a}{\lambda}}.$$

12. Prove that the perimeter and area of a single loop of the curve

$$r = 2a\cos n\theta \quad (n > 1)$$

are respectively the same as those of an ellipse of semi-axes a and $\dfrac{a}{n}$.

XCVIII.

1. Prove that, if two opposite sides of a quadrilateral inscribed in a given circle touch another given circle, the other sides of the quadrilateral touch a third circle co-axial with the first two.

2. Prove that the common chord of a conic and its circle of curvature at any point and the tangent at the point divide their other common tangent harmonically.

3. Prove that the necessary and sufficient condition that

$$\alpha\,.\,\Sigma x^3 + \beta\,.\,\Sigma x^2 y + \gamma\,.\,\Sigma xy^2 + \delta xyz = 0 \quad (\beta^3 \neq \gamma^3)$$

should be the product of three linear factors is

$$9\alpha^3 - 3\alpha^2\delta - \alpha\,(3\beta\gamma - \delta^2) + \beta^3 + \gamma^3 - \beta\gamma\delta = 0.$$

4. Prove that

$$\frac{1}{1\,.\,3}\cdot\frac{1}{2^3} + \frac{1}{3\,.\,5}\cdot\frac{1}{2^5} + \frac{1}{5\,.\,7}\cdot\frac{1}{2^7} + \dots \text{ ad inf.} = \tfrac{1}{4} - \tfrac{3}{16}\log 3.$$

5. Prove that, if a and c have the same sign, and

$$\frac{1}{a + bxy + cy^2} = P_0 + P_1 y + P_2 y^2 + \dots,$$

so that P_n is a function of x, then the equation $P_n = 0$ has all its roots real.

6. If Δ, Δ' are the areas of two triangles whose sides satisfy one of the relations

$$\sqrt{a^2 - a'^2} \pm \sqrt{b^2 - b'^2} \pm \sqrt{c^2 - c'^2} = 0,$$

then

$$\sqrt{a^2\Delta'^2 - a'^2\Delta^2} \pm \sqrt{b^2\Delta'^2 - b'^2\Delta^2} \pm \sqrt{c^2\Delta'^2 - c'^2\Delta^2} = 0.$$

7. Prove that

$$x + \frac{x^7}{7} + \frac{x^{13}}{13} + \frac{x^{19}}{19} + \dots \text{ ad inf.}$$

$$= \frac{1}{12}\log\frac{(1+x)^2(1+x+x^2)}{(1-x)^2(1-x+x^2)} + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$$

8. Prove that the equation to the circle of curvature at the point a of the conic

$$\frac{l}{r} = 1 + e\cos\theta$$

is $$(1 + 2e\cos a + e^2)\left\{1 - \left(\frac{l}{r} - e\cos\theta\right)^2\right\}$$
$$+ e^2\left\{\frac{l}{r} - e\cos\theta - \cos(\theta - a)\right\}\left\{\frac{l}{r}\cdot\frac{e\cos a + \cos 2a}{1 + e\cos a}\right.$$
$$\left. - e\cos\theta - \cos(\theta + a)\right\} = 0.$$

9. Shew that the locus of the foci of rectangular hyperbolas for which the triangle of reference is self-conjugate is

$$\Sigma\left\{a^2(-\alpha\cos A + \beta\cos B + \gamma\cos C) + \alpha\beta\gamma\right\}^{-1} = 0.$$

10. Two particles of masses m and m' connected by a string of length l rest on a smooth cycloid with its vertex upwards and base horizontal. Prove that in the equilibrium position the distance of the particle m from the vertex measured along the arc is

$$\frac{m'}{m + m'}\cdot l.$$

11. A hollow elliptic cylinder is placed so that its length is horizontal. A section perpendicular to its length has its major axis vertical and of length $2a$, and the eccentricity is e. A particle is projected from the lowest point. If it makes complete revolutions shew that the initial velocity must be at least

$$\{ga(5 - e^2)\}^{\frac{1}{2}}.$$

12. If r be a focal radius vector of any point on an ellipse, p the perpendicular from the focus on the tangent at the point, and $2l$ the latus-rectum of the ellipse, prove that the value of the integral

$$\int\frac{p\,ds}{r^3}$$

taken round the ellipse is $2\pi/l$.

XCIX.

1. Circles are described about the four triangles formed by the sides of a quadrilateral, which itself circumscribes a circle. Prove that of the centres of similitude of the six pairs of these circles, six lie on a straight line.

2. Three conics α, β, γ have a common focus, and are such that β and γ touch at P, γ and α touch at Q, α and β touch at R. Shew that the tangents at P, Q and R meet the corresponding directrices of α, β and γ respectively in three collinear points.

3. If a, $ab-h^2$ and $abc+2fgh-af^2-bg^2-ch^2$ are all positive, shew that the expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

is positive for all real values of x, y and z.

4. If n is an integer such that $\frac{1}{3}(n^2-1)$ is a perfect square, then n must be included in the formula

$$2^m+3\,.\,{}^mC_2\,.\,2^{m-2}+3^2\,.\,{}^mC_4\,.\,2^{m-4}+\dots$$

where m is any integer.

5. If the equations

$$ax^3+3bx^2+3cx+d=0, \quad a'x^3+3b'x^2+3c'x+d'=0$$

have two common roots, then the equation in λ, viz.:

$$\begin{vmatrix} a+\lambda a', & 2(b+\lambda b'), & c+\lambda c', & 0 \\ 0, & a+\lambda a', & 2(b+\lambda b'), & c+\lambda c' \\ b+\lambda b', & 2(c+\lambda c'), & d+\lambda d', & 0 \\ 0, & b+\lambda b', & 2(c+\lambda c'), & d+\lambda d' \end{vmatrix}=0,$$

must have two pairs of equal roots.

6. If
$$x\sec\alpha+y\operatorname{cosec}\alpha+z\tan\alpha=2\operatorname{cosec}2\alpha,$$
$$x\sec\beta+y\operatorname{cosec}\beta+z\tan\beta=2\operatorname{cosec}2\beta,$$
$$x\sec\gamma+y\operatorname{cosec}\gamma+z\tan\gamma=2\operatorname{cosec}2\gamma,$$

where α, β, γ are unequal and less than π, then for all values of θ,

$$x\sin\theta+y\cos\theta+z\sin^2\theta$$
$$=1-4z\sin\tfrac{1}{2}(\theta-\alpha)\sin\tfrac{1}{2}(\theta-\beta)\sin\tfrac{1}{2}(\theta-\gamma)\sin\tfrac{1}{2}(\theta+\alpha+\beta+\gamma).$$

7. Prove that

$$\frac{\cosh \pi - \cos \pi \sqrt{3}}{\cosh \pi \sqrt{2} - \cos \pi \sqrt{2}} = \prod_{1}^{\infty} \left\{1 - \frac{1}{n^2 + n^{-2}}\right\}.$$

8. Shew that the line $lx + my = 1$ cuts the conic

$$ax^2 + 2hxy + by^2 = 1$$

at angles given by

$$\left[(al^2 + 2hlm + bm^2) S' - (l^2 + m^2)^2\right] \tan^2 \theta$$
$$+ 2S' \left[(a-b) lm - h (l^2 - m^2)\right] \tan \theta + S' (am^2 - 2hlm + bl^2) = 0,$$

where $$S' = \frac{am^2 - 2hlm + bl^2}{ab - h^2} - 1.$$

9. S and S' are any two conics, and the tangent at any point of S meets the polar of that point with regard to S' in Q. Prove that the locus of Q is a quartic curve, having double points at the angular points of the common self-conjugate triangle of S and S'.

10. A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if

$$R/r > \sqrt{5} - 1,$$

the equilibrium is stable, whatever be the weight of the particle.

11. A cylindrical block of mass M resting on a smooth horizontal table has a plane base, and the cross-section is a cycloid. A particle of mass m is placed at the centre of the highest generator, and is slightly displaced in a direction perpendicular to the generator. Shew that the particle will leave the cylinder at a point P given by

$$\tan^4 \phi = \frac{M}{M+m},$$

where ϕ is the angle made by the tangent at P with the horizon.

12. Prove that if a rectangular hyperbola be drawn having four-point contact with a curve at a point where the radius of curvature and its differential coefficient with respect to the arc are respectively ρ and ρ', the length of the transverse axis of the hyperbola is

$$2\rho/(1 + \tfrac{1}{9} \rho'^2)^{\frac{3}{4}}.$$

C.

1. If $O(AA', BB', CC')$ be any pencil in involution and ABC, $A'B'C'$ any two triangles with their angular points on the rays, then the rays OA, OB, OC cut $B'C'$, $C'A'$, $A'B'$ respectively in collinear points, and the rays OA', OB', OC' cut BC, CA, AB respectively in collinear points.

2. Prove that the director circles of all conics touching the two straight lines OP, OQ at P and Q have as a common radical axis, the radical axis of O and the circle on PQ as diameter.

3. If $x_1, x_2, x_3 \ldots x_n$ be any n positive quantities, shew that

$$3(n-2)\,.\,\Sigma x_1\,.\,\Sigma x_1^2 + \tfrac{1}{2}(n-2)(n-7)\,.\,\Sigma x_1^3 \not< 21\,.\,\Sigma x_1 x_2 x_3.$$

4. Prove that the square of the product of all numbers prime to, and not greater than, any given integer N is, when diminished by unity, a multiple of N.

5. If the coefficients in the equation

$$x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

are real, and the roots are all complex with equal moduli, then

$$e^3 = a^3 f^2, \quad de = abf$$

and the real parts of the roots are α, β, γ, where 2α, 2β, 2γ are the roots of the equation

$$y^3 + ay^2 + \left(b - \frac{3af}{e}\right) y + c - \frac{2a^2 f}{e} = 0.$$

6. A, B, C are the centres of three circles of radii r_1, r_2, r_3 and the radius of the circle cutting the three circles orthogonally is ρ. Shew that

$$16\Delta^2\rho^2 = \Sigma a^2 r_1^4 - 2\Sigma r_2^2 r_3^2\, bc \cos A - 2abc\, \Sigma r_1^2 a \cos A + a^2 b^2 c^2,$$

where Δ, a, A, etc. refer to the triangle ABC.

7. ONP_1 is a triangle, right-angled at N. The side NP_1 is produced and on it points P_2, P_3, $P_4 \ldots$ are taken such that

$$NP_r = (2r-1)^2\,.\,NP_1.$$

Prove that

$$\frac{OP_1\,.\,OP_2\,.\,OP_3 \ldots}{NP_1\,.\,NP_2\,.\,NP_3 \ldots} = \frac{1}{\sqrt{2}} \left\{ \cosh \sqrt{\frac{ON}{2NP_1}}\,.\,\pi + \cos \sqrt{\frac{ON}{2NP_1}}\,.\,\pi \right\}^{\frac{1}{2}}.$$

8. Shew that the equation to the circle circumscribing the triangle formed by the three lines

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \quad a_3x + b_3y + c_3 = 0,$$

is
$$\begin{aligned}(x^2 + y^2) \,.\, \Pi\, (a_2b_3 - a_3b_2) & \\ + x \,.\, \Sigma c_1 \,(a_2b_3 - a_3b_2)\,(a_1a_2a_3 - a_1b_2b_3 + a_2b_3b_1 + a_3b_1b_2) & \\ + y \,.\, \Sigma c_1 \,(a_2b_3 - a_3b_2)\,(b_1b_2b_3 - b_1a_2a_3 + b_2a_3a_1 + b_3a_1a_2) & \\ - \Sigma c_2c_3\,(a_1{}^2 + b_1{}^2)\,(a_2b_3 - a_3b_2) = 0. &\end{aligned}$$

9. Prove that the origin is a centre of curvature of the general conic if
$$a + b - c\rho^{-2} = -3\Delta^{\frac{1}{3}}\rho^{-\frac{2}{3}}$$
and
$$a' + b' - c'\rho^2 = -3\Delta^{-\frac{1}{3}}\rho^{\frac{2}{3}},$$
where ρ is the corresponding radius of curvature, Δ the discriminant and $a' = \dfrac{1}{\Delta}\dfrac{\partial\Delta}{\partial a}$, etc.

10. Four similar rods, each of length a and mass m, have their ends jointed so as to form a square $ABCD$; to the corners of this framework are attached eight rods EA, EB, EC, ED and FA, FB, FC, FD similar in all respects to the former ones. Shew that if all the joints are smooth, and if a sphere of radius c and mass M rests upon the four lower rods when the system is suspended from E, the stress in one of the horizontal rods is

$$\frac{g}{a\sqrt{2}}\,\{6ma + M\,(a - c)\}.$$

11. An inelastic string passes over a small smooth pulley and hangs vertically on either side of it. To its two ends are attached masses M_1 and M_2 $(M_1 > M_2)$ which are also attached to the ends of an elastic string. Originally M_1 and M_2 are on the same level and the string is therefore slack. If the system now moves from rest, find the greatest depth to which M_1 sinks, and shew that when the elastic string is tight M_1 and M_2 move in simple harmonic motion of period

$$\pi\sqrt{\frac{l_0\,(M_1 + M_2)}{E}},$$

where l_0 is the unstretched length and E the modulus of elasticity.

12. Prove that the normal to the curve $r^2 = a^2\cos 2\theta$ at the point $\theta = \alpha$, meets the perpendicular normal at a point whose distance from the origin is

$$2^{-1} \,.\, 3^{\frac{1}{4}} \,.\, a\cos^{\frac{1}{2}}\left(2\alpha \pm \frac{\pi}{6}\right).$$

Cambridge:
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS